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ERRATUM.

Page 217, line 9 from the bottom. *For* ε *read* ε^{jPt}.

methods of arriving at the chief formula and objective means of illustrating the results, so as to give the student confidence in the mathematical deductions. I have found the following method of discussing the phenomena well suited for engineering students, as it makes use of ideas brought before them in other studies and involves no mathematics beyond the simple algebra of complex quantities.

We may discuss as follows first the problem of a condenser discharging through an inductive circuit. To make the problem general, we shall assume the condenser to be a leaky one.

Suppose free damped oscillations have been established in a circuit consisting of a wire having resistance R and inductance L, completing the circuit of a condenser of capacity C having leakance S. Let n be the frequency of these oscillations, and let $2\pi n=p$. If, then, we assume, as we may do, that the currents and voltages vary harmonically, and are proportional to the real part or horizontal step of ε , where $j=\sqrt{-1}$ and $P=p+j\alpha$, in which α is the damping factor, then the voltage drop down the resistance at the instant when the current is I is $(R+jPL)I$. Also the voltage drop down the condenser is $\frac{I}{S+jPC}$, and, since the oscillations are free, there is no impressed E.M.F. Hence the total voltage drop down the whole circuit is zero, or

$$(R+jPL)+\frac{1}{S+jPC}=0. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Therefore $$(R+jPL)(S+jPC)=-1=j^2. \quad . \quad . \quad . \quad . \quad (2)$$

Divide all through by CL, and put $2a$ for R/L and $2b$ for S/C. Then

$$(jP+2a)(jP+2b)=\frac{j^2}{CL}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Add to both sides $(a-b)^2$ and rearrange, and we have

$$\left\{ jP+(a+b)\right\}^2=j^2\left\{\frac{1}{CL}-(a-b)^2\right\}. \quad . \quad . \quad . \quad (4)$$

Taking the square root and multiplying by $-j$ all through we have

$$P=p+ja=j(a+b)\pm\sqrt{\frac{1}{CL}-(a-b)^2}, \quad . \quad . \quad . \quad (5)$$

and then equating real parts we have

$$p=2\pi n=\pm\sqrt{\frac{1}{CL}-\left(\frac{R}{2L}-\frac{S}{2C}\right)^2}, \quad . \quad . \quad . \quad (6)$$

which gives us the frequency of the oscillations in the circuit. The damping factor $\alpha=(a+b)$ and the ratio of two successive oscillations is $\varepsilon^{-(a+b)T/2}$; hence the decrement δ of the oscillations per half period is

$$(a+b)T/2, \text{ or } \delta=\frac{1}{2n}\left(\frac{R}{2L}+\frac{S}{2C}\right) \quad . \quad . \quad . \quad . \quad (7)$$

If the condenser is assumed to be non-leaky, we have the usual formulæ for the frequency and decrement of a non-radiative or closed circuit thus arrived at in the simplest manner. The equation (6) shows that the frequency depends on the damping and that the rate at which the oscillations decay increases as the resistance R of the circuit or the leakance S of the condenser increases. These formulæ can be confirmed both qualitatively and quantitatively by experiments made with a Duddell oscillograph. For this purpose we attach to the shaft of an ordinary single-phase alternator a commutator consisting of brass segments let into an insulating disk of stabilit, one sector for each pole of the alternator. Three metal gauze brushes press against the sectors and the sectors are staggered, so that whilst the middle brush always rubs on a sector the outside brushes are alternately in metallic contact with the middle brush as the sectors run round. The motor of the oscillograph is driven synchronously direct off the alternator. A condenser of suitable capacity has one terminal

connected to the middle brush, and the two outside brushes are connected respectively to a battery of cells and to the condenser discharge circuit. The remaining terminals of the condenser, battery and discharge circuit are connected together. The condenser is then charged and discharged synchronously with the movement of the tilting mirror of the oscillograph. The oscillograph wires can be connected through resistances with the terminals of the condenser or with two points on the discharge circuit.

We have then on the oscillograph screen a visible representation of the oscillation of current or condenser P.D., which can be photographed. The oscillograms in the appended Plate were taken with a condenser of the capacity of about 5 mfd. and with variable resistance in the discharge circuit. They show the slight variation of frequency with resistance, and the rapidly increasing decrement at the resistance R increases. In this case the condenser was non-leaky. The critical case, when $\frac{1}{CL}=\frac{R^2}{4L^2}$, corresponding to cessation of oscillations, is confirmed by the oscillograms Nos. 11, 12, and 13. The value of R, which just annuls oscillation, agrees closely with the value predicted by theory.

Supposing the resistance of the discharge circuit is kept constant and small, so that $R^2/4L^2$ is small, and if the leakance of the condenser is gradually increased by a non-inductive resistance placed as a shunt across the terminals, then there should also be a decrement which increases with the value of S, and finally extinguishes the oscillations when

$$\frac{1}{CL}=\left(\frac{R}{2L}-\frac{S}{2C}\right)^2.$$

This is well shown by the oscillograms Nos. 14 to 21, which illustrate the effect of gradually increasing the leakance of the condenser, and show that the leakance, which just annuls all oscillations, agrees with that predicted by theory.

Turning, then, to the case of coupled oscillatory circuits, we can establish in the same simple manner the essential theory of the interaction.

Let there be two oscillatory circuits having condensers of capacity C_1 and C_2, and leakances S_1 and S_2, in series with circuits of resistance R_1 and R_2 and inductance L_1 and L_2 respectively. Let free oscillations be excited in one circuit and react upon the other, the mutual inductance being M.

Then if these oscillations are damped they can be represented by the real parts of $\varepsilon^{j(p+ja)t}=\varepsilon^{jPt}$, where $p=2\pi$ times the frequency and a is the damping coefficient$=R/2L+S/2C$. Hence, when the circuits are in oscillation and left to themselves, the currents I_1 and I_2 are determined by the two equations

$$\left.\begin{aligned}\left\{(R_1+jP_1L_1)+\frac{1}{S_1+jP_1C_1}\right\}I_1+jMP_2I_2=0,\\ jMP_1I_1+\left\{(R_2+jP_2L_2)+\frac{1}{S_2+jP_2C_2}\right\}I_2=0.\end{aligned}\right\} \quad . \quad . \quad (8)$$

Writing Tim_1 for the function $R_1+jP_1L_1+(S_1+jP_1C_1)^{-1}$ and Tim_2 for the same function for the secondary circuit, these being respectively the total impedances for damped oscillations, we have

$$\left.\begin{aligned}Tim_1I_1+jMP_2I_2=0,\\ jMP_1I_1+Tim_2I_2=0.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Eliminating I_1 and I_2 the determinant

$$\begin{vmatrix} Tim_1, & jMP_2 \\ jMP_1, & Tim_2 \end{vmatrix}=0, \quad . \quad . \quad . \quad . \quad (10)$$

or $Tim_1\ Tim_2+M^2P_1P_2=0$ gives us an equation determining the frequency of the oscillations set up in the circuits.

Suppose we limit ourselves to the case of non-leaky condensers and consider the resistance of the secondary circuit to be small. Then $S_1=S_2=0$ and also if $R_2=0$, we have $P_2=p=2\pi n$.

Let us denote the mutual inductance by $M=k\sqrt{L_1L_2}$, where k is the coefficient of coupling; then the determinant (10) reduces to

$$(1-P_1^2C_1L_1+jP_1C_1R_1)(1-P_2^2C_2L_2+jP_2C_2R_2) \\ =M^2P_1^2P_2^2C_1C_2, \quad (11)$$

or, since $P_2=p$ and $P_1=p+ja_1$ and $M^2=k^2L_1L_2$, we have

$$(1-p^2C_2L_2)\left[1-(p^2-a_1^2)C_1L_1-j2a_1pC_1L_1+jpC_1R_1-a_1C_1R_1\right] \\ =k^2(p^2-a_1^2)p^2C_1L_1C_2L_2+jk^22a_1p^3C_1L_1C_2L_2.$$

Equating real parts of the above equation and remembering that $a_1=R_1/2L_1$, we find at once on substitution that

$$1-p^2(C_1L+C_2L_2)+p^4(1-k^2)C_1L_1C_2L_2 \\ =a_1^2(C_1L_1-k^2p^2C_1L_1C_2L_2-p^2C_1L_1C_2L_2). \quad . \quad (12)$$

If we assume the circuits are tuned, or that $C_1L_1=C_2L_2=CL$, then equation (12) reduces to

$$\left(p^2CL(1-k)-1\right)\left(p^2CL(1+k)-1\right) - a_1{}^2CL(1-(1+k^2)p^2CL)=0. \quad . \quad (13)$$

Suppose, then, that a_1 is very small, or that the damping in the primary is negligible, the equation (13) is seen to be satisfied by

$$p=\frac{1}{\sqrt{CL(1-k)}} \text{ and by } p=\frac{1}{\sqrt{CL(1+k)}},$$

and there are, therefore, oscillations of two frequencies given by the above well-known formulæ. If, however, a_1 is very large, or the primary current rapidly quenched by increasing the resistance R_1 to infinity, then the coupling becomes zero when the primary circuit is open, and we have only one oscillation, which is the free oscillation of the secondary circuit. For intermediate values of a there will be three frequencies present. These well-known results are illustrated in the oscillograms Nos. 23 to 40 in the appended Plates which show the effect of gradually increasing the coupling of two inductively coupled circuits, and thus creating in the oscillation trains *beats* indicating the presence of two frequencies. The primary circuit in this case remains closed. If the primary circuit is opened very soon, so as to quench the primary current, then the oscillograms Nos. 32 to 36 show the existence of only a single-period oscillation, which is the free oscillation of the secondary circuit, provided the coupling is not very close. The oscillograms show that even if the two circuits are not tuned when the primary current is suddenly arrested, we have in the secondary an oscillation of frequency given by $p=2\pi n=1/\sqrt{C_2L_2}$, as it should be.

In the case when the coupling is not very loose and the primary current not instantly damped, we may have oscillations of three periods co-existing in the secondary circuit one, its own free oscillation corresponding to the last factor in equation (13), and the other two the periods due to the reaction of the two circuits prior to the instant when the primary resistance becomes infinite. This is well shown in a number of resonance curves described by the Author and Mr. Dyke in a Paper on "Some Resonance Curves taken with Impact and Spark Ball Discharges," "Proc." Phys. Soc., Lond.,

Vol. XXIII., 1911, p. 136, in which resonance curves with three humps are shown. The function *Tim* above defined is, of course, only an extended form of Heaviside's resistance operator, the application of which in all problems involving oscillation transformers was recently explained by Dr. Eccles in a Paper on the "Application of Heaviside's Resistance Operators to the Theory of the Air-core Transformer and Coupled Circuits in General." (*See* W. H. Eccles, D.Sc., "Proc." Phys. Soc., Lond., Vol. XXIV., 1912, p. 261.)

The function *Tim* may be called the complete resistance operator of a circuit having inductance and capacity, and the whole theory of the transformer is then embraced in the equations (8), (9) and (10). The application to predict the graph of the primary and secondary current is given in Dr. Eccles' Paper above. The oscillograms given in connection with the present Paper have been carefully taken by Mr. Dyke, and as the capacities, inductances and resistances are given in every case, they will serve to check the theory and to enable any student to predetermine the frequencies which will arise without the necessity for anything but the simplest algebraic equations. It is possible from the oscillograms, by measuring the ratio of successive oscillations, to make a fairly good approximation to the calculated decrement from the known resistance, capacity and inductance of the circuit.

The above Paper was illustrated by about 40 oscillograms of condenser discharges of single and coupled circuits prepared as lantern slides.*

They were as follows (*see* Plate) :—

Series I. (*Slides 1 to 5*).—The first set of five slides show the oscillatory discharge of condensers having capacities respectively of 4·0, 2·0, 1·0, 0·5, 0·25 mfd. when discharged with oscillations through an inductive resistance having an inductance of 0·031 henry and a resistance less than 19 ohms. The frequencies corresponding to these various capacities are 450, 640, 900, 1,280 and 1,800, and measurements made on these oscillograms show that the frequency varies inversely as the square root of the capacity, as it should do by theory.

Series II. (*Slides 6 to 10*).—The second set of five slides show the effect of increasing the resistance of the charge circuit, the

* The Cambridge Scientific Instrument Co., of Cambridge, England, have undertaken to reproduce and supply these slides to colleges and teachers who may be desirous of possessing them, but have not the means of preparing the oscillographs themselves.

capacity remaining constant at 5·0 mfd. and the inductance at 0·031 henry. The resistance is successively increased to the values 4·4, 22·3, 51·6, 72·0 and 112·0 ohms, and the effect of this is shown by the progressively increasing damping of the oscillations, the frequency remaining constant.

It should be noted that in the first two slides of this series the particular arrangement used for obtaining the oscillogram does not permit the whole of the oscillations in the train to be photographed ; only three or four complete oscillations are shown, but this does not exhaust the energy stored up. It is, however, sufficient to show the damping or ratio of one oscillation to the next.

The decrement δ in each case can be calculated from the formula $\delta = R/4nL$, and can be measured from the ratio of successive amplitudes, since $\delta = \log_e I_1/I_2$.

Corresponding to the various resistance (R) and frequencies (N) the calculated decrements and decrements measured from the oscillograms are as follows :—

R.	N.	δ (calc.).	δ (obs.).
4·4	405	0·07	0·11
22·0	400	0·45	0·38
51·6	363	1·05	1·06
72·0	356	1·63	1·6
112·0	284	3·15	—

Series II.a (*Slides 11 to 13*).—Three slides, forming a continuation of Series II. on an enlarged scale. The well-known formula for the frequency

$$N = \frac{1}{2\pi}\sqrt{\frac{1}{CL} - \frac{R^2}{4L^2}}$$

shows that in the case of a condenser discharged the discharge ceases to be oscillatory when $\frac{1}{CL} = \frac{R^2}{4L^2}$.

The three slides in this series show the observed effects when $\frac{1}{CL}$ is a little greater than, equal to, in a little less than $\frac{R^2}{4L^2}$.

In a condenser circuit the damping depends not only on the resistance of the circuit in series with the condenser, but on any leak in the condenser.

Series II.b (*Slides 14 to 18*).—Consists of five slides showing the effect of gradually diminishing a resistance (r) in parallel with the condenser. In this case the capacity was 5·0 mfd., the inductance of the circuit 0·031 henry and the series resistance about 4 ohms. The shunt resistance was made suc-

cessively infinity, 278, 120, 86 and 55 ohms, and the oscillograms show the gradually increasing damping. Corresponding to this, the frequency (N) and the observed and calculated decrements are as given below :—

r.	N.	δ (calc.).	δ (obs.).
00	405	0·08	0·07
278	400	0·53	0·57
120	382	1·17	1·24
86	356	1·71	1·63
55	284	3·26	—

Series II.c (*Slides 19 to 22*).—This consists of four slides which are a continuation of Series II.b on an enlarged scale, intended to show that for the shunt conductivity S the discharge just ceases to be oscillatory when $\frac{1}{LC}=\frac{S^2}{4C^2}$. Three of the oscillograms show the state of affairs when $\frac{1}{LC}$ is a little greater than, equal to, or a little less than $\frac{S^2}{4C^2}$, and the fourth when $\frac{1}{\sqrt{LC}}=\frac{R}{2L}=\frac{S}{2C}$, when the discharge again becomes oscillatory, though the damping is large.

Series III. (*Slides 23 to 26*).—Four oscillograms illustrate the reaction between a primary and secondary circuit, both having capacity and inductance, but coupled with various degrees of coupling (mutual inductance).

The primary and secondary circuits are tuned so that the product CL is the same for both, and the frequency is 1,610. It is well known that when the coupling is close (as shown on slide 26), then the reaction of the circuits produces in each oscillations of two different frequencies, and these make themselves evident by producing the phenomena of *beats* in the train of oscillations.

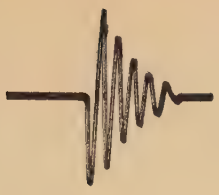
Series IV. (*Slides 27 to 28*).—These two oscillograms are taken with the same coupled circuits, only in the first case the primary current is not quenched, and in the second it is rapidly damped out to show the resulting annihilation of the *beats*.

Series V. (*Slides 29 to 34*).—These six oscillograms of the oscillations in two coupled circuits illustrate the effect of varying the resistance, and therefore damping in the two circuits.

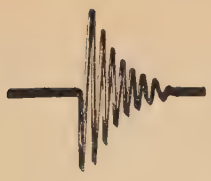
The circuits were tuned and loosely coupled. The slides 29, 30, 31 show the effect of gradually increasing the resistance

PLATE.

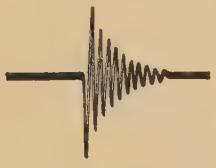
(1) Series I. L=0·031 henry. R<19 ohms C=4·0 mfds. n=450.



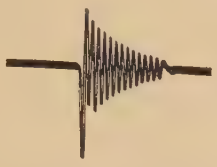
(2) Series I. L=0·031 henry. R<19 ohms C=2·0 mfds. n=640.



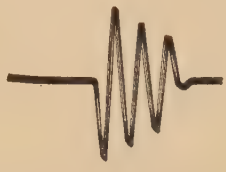
(3) Series I. L=0·031 henry. R<19 ohms C=1·0 mfds. n=900.



(4) Series I. L=0·031 henry. R<19 ohms C=0·5 mfd. n=1280.

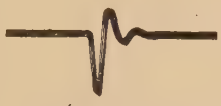


(5) Series I. L=0·031 henry. R<19 ohms C=0·25 mfd. n=1800.

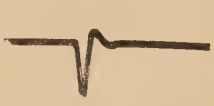


(6) Series II. L=0·031 henry. C=5·0 mfds. R=4·4 ohms.

(7) Series II. L=0·031 henry. C=5·0 mfds. R=22·3 ohms.



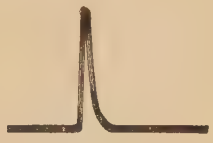
(8) Series II. L=0·031 henry. C=5·0 mfds. R=51·6 ohms.



(9) Series II. L=0·031 henry. C=5·0 mfds. R=72 ohms.

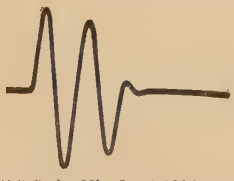
(10) Series II. L=0·031 henry. C=5·0 mfds. R=112 ohms.

(11) Series IIa. L=0·031 hny. C=5·0 mfds. R=112 ohms.
$\frac{1}{LC} > \frac{R^2}{4L^2}$.

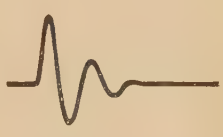


(12) Series IIa. L=0·031 hny. C=5·0 mfds. R=155 ohms.
$\frac{1}{LC} = \frac{R^2}{4L^2}$.

(13) Series IIa. L=0·031 hny. C=5·0 mfds. R=400 ohms.
$\frac{1}{LC} < \frac{R^2}{4L^2}$.



(14) Series IIb. L=0·031 hny. C=5·0 mfds. R=4·0 ohms. Condenser shunt=∞ ohms.



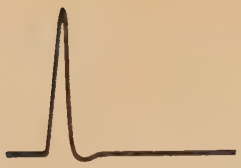
(15) Series IIb. L=0·031 hny. C=5·0 mfds. R=4·0 ohms. Condenser shunt 278 ohms.

(16) Series IIb. L=0·031 hny. C=5·0 mfds. R=4·0 ohms. Condenser shunt=120 ohms.

(17) Series IIb. L=0·031 hny. C=5·0 mfds. R=4·0 ohms. Condenser shunt=86 ohms.

(18) Series IIb. L=0·031 henry. C=5·0 mfds. R=4·0 ohms. Condenser shunt=55 ohms.

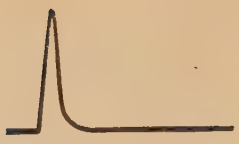
To face p. 224.]



(19) Series IIc. L=0·031 hny. C=5·0 mfds. R=7·0 ohms.

$\frac{1}{S}$=Condenser Shunt =55 ohms.

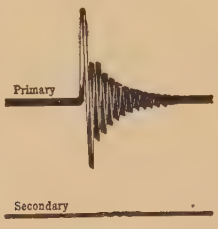
$$\frac{1}{LC} > \frac{S^2}{4C^2}.$$



(20) Series IIc. L=0·031 hny. C=5·0 mfds. R=7·0 ohms.

$\frac{1}{S}$=Condenser Shunt =40 ohms.

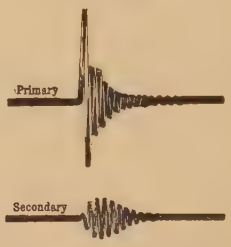
$$\frac{1}{LC} = \frac{S^2}{4C^2}.$$



(21) Series IIc. L=0·031 hny. C=5·0 mfds. R=7·0 ohms.

$\frac{1}{S}$=Condenser Shunt =15·5 ohms.

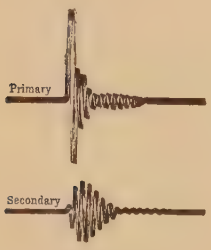
$$\frac{1}{LC} < \frac{S^2}{4C^2}.$$



(22) Series IIc. L=0·031 hny. C=5·0 mfds. R=155 ohms.

$\frac{1}{S}$=Condenser Shunt =40 ohms.

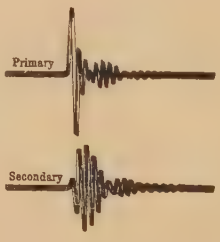
$$\frac{1}{LC} = \frac{R^2}{4L^2} + \frac{S^2}{4C^2}.$$



(23) Series III. Zero coupling.

Prim. { L=0·013 henry. C=0·75 mfd. R=15·6 ohms.

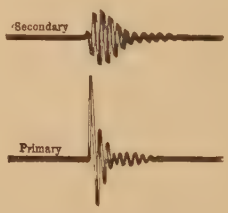
Sec. { L=0·013 henry. C=0·75 mfd. R=16·5 ohms.



(24) Series III. Loose coupling.

Prim. { L=0·013 henry. C=0·75 mfd. R=15·6 ohms.

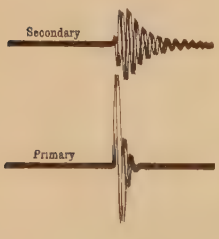
Sec. { L=0·013 henry. C=0·75 mfd. R=16·5 ohms.



(25) Series III. Moderate Coupling.

Prim. { L=0·013 henry. C=0·75 mfd. R=15·6 ohms.

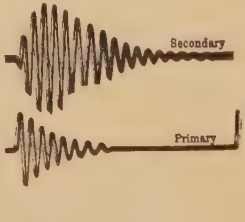
Sec. { L=0·013 henry. C=0·75 mfd. R=16·5 ohms.



(26) Series III. Close Coupling.

Prim. { L=0·013 henry. C=0·75 mfd. R=15·6 ohms.

Sec. { L=0·013 henry. C=0·75 mfd. R=16·5 ohms.

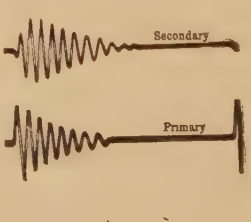


(27) Series IV. Moderate Coupling.

Sec. { L=0·013 henry. C=0·75 mfd. R=16·5 ohms.

Prim. { L=0·013 henry. C=0·75 mfd. R=15·6 ohms.

Primary not quenched.



(28) Series IV. Moderate Coupling.

Sec. { L=0·013 henry. C=0·75 mfd. R=16·5 ohms.

Prim. { L=0·013 henry. C=0·75 mfd. R=15·6 ohms.

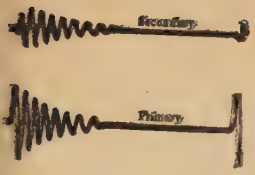
Primary quenched at first. Current node.

(29) Series V.

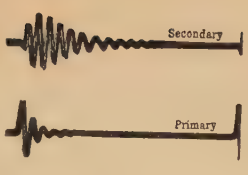
Pr. C=2 mfds. L=0·013 hny.
Sec. C=2 mfds. L=0·013 hny.
Sec. R=11·5 ohms.
Pr. R= 5·4 ohms.

(30) Series V.

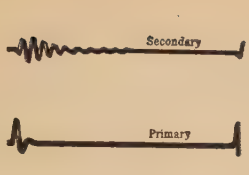
Pr. C=2 mfds. L=0·013 hny.
Sec. C=2 mfds. L=0·013 hny.
Sec. R=27 ohms.
Prim. R=5·4 ohms.



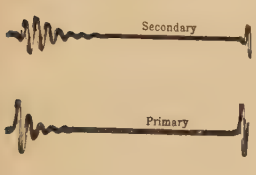
(31) Series V.
Pr. C=2 mfds. L=0·013 hny.
Sec. C=2 mfds. L=0·013 hny.
Sec. R=53 ohms.
Pr. R=5·4 ohms.



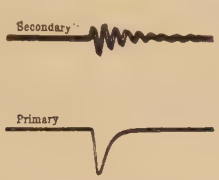
(32) Series V.
Pr. C=2 mfds. L=0·013 hny.
Sec. C=2 mfds. L=0·013 hny.
Sec. R=11·5 ohms.
Prim. R=24·9 ohms.



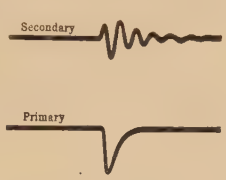
(33) Series V.
Pr. C=2 mfds. L=0·013 hny.
Sec. C=2 mfds. L=0·013 hny.
Sec. R=11·5 ohms.
Prim. R=88 ohms.



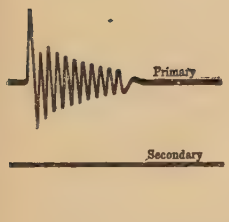
(34) Series V.
Pr. C=2 mfds. L=0·013 hny.
Sec. C=2 mfds. L=0·013 hny.
Sec. R=27 ohms.
Pr. R=24·9 ohms.



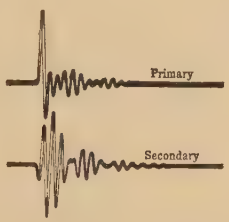
(35) Series VI. Aperiodic Primary, Secondary Tuned.
Sec. { L=0·013 henry. C=2·0 mfds. R=11·5 ohms.
Prim. { L=0·013 henry. C=2·0 mfds. R=462 ohms.



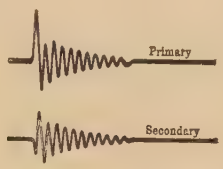
(36) Series VI. Aperiodic Primary, Secondary not Tuned.
Sec. { L=0·013 henry. C=4·0 mfds. R=11·5 ohms.
Prim. { L=0·013 henry. C=2·0 mfds. R=462 ohms.



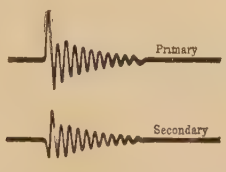
(37) Series VII. Zero coupling Circuits Tuned.
Prim. { L=0·013 henry. C=0·5 mfd. R=6·0 ohms.
Sec. { L=0·013 henry. C=0·5 mfd. R=28 ohms.



(38) Series VII. Close coupling Circuits Tuned.
Prim. { L=0·013 henry. C=0·5 mfd. R=6·0 ohms.
Sec. { L=0·013 henry. C=0·5 mfd. R=28 ohms.



(39) Series VII. Close coupling Circuits Tuned ($C_1L_1=C_2L_2$) Secondary Aperiodic.
Prim. { L=0·013 henry. C=0·5 mfd. R=6·0 ohms.
Sec. { L=0·013 ohms. C=0·5 mfd. R=400 ohms.



(40) Series VII. Close coupling Circuits not Tuned ($C_1L_1 \ddagger C_2L_2$) Secondary Aperiodic.
Prim. { L=0·013 henry. C=0·5 mfd. R=6·0 ohms.
Sec. { L=0·013 henry C=4·0 mfds. R=400 ohms.

in the secondary circuit, whilst that in the primary remains constant and at low value. The primary resistance = 5·4 ohms and the secondary resistance = 11·5, 27 and 53 ohms. It will be noticed that the secondary never ceases to oscillate before the primary.

The slides 32, 33, 34 show the effect of gradually increasing the resistance in the primary, whilst that in the secondary remains constant.

Series VI. (*Slides 35 to 36*).—In these cases the damping of the primary is increased until it becomes aperiodic, and it is then seen that the oscillations in the secondary are solely determined by the capacity and inductance of the secondary circuit, no matter whether the circuits are tuned or not. In slide 35 the primary and secondary circuits are tuned, and in slide 36 they are not tuned, but the frequency in the secondary is the same—viz., its own free natural frequency in the two cases.

Series VII. (*Slides 37 to 40*).—This comprises four oscillograms showing the effect of an aperiodic secondary circuit. Slide 37 shows the primary oscillations when the secondary is open, slide 38 the result of closely coupling a tuned secondary circuit of small damping. Slides 39 and 40 show the result of increasing the secondary resistance until it becomes a forced oscillation of the same frequency as the primary, no matter how greatly the secondary capacity is varied. The scale of the last two slides is different from that of Nos. 37 and 38.

In conclusion, the Author desires to thank his former assistant, Mr. G. B. Dyke, B.Sc., who skilfully photographed all the above-mentioned oscillograms, and prepared therefrom the lantern slides, which have been reduced in scale for the accompanying Plate.

ABSTRACT.

The author gave a very short method, involving only the simplest algebra, for arriving at a formula for the time of free electrical oscillation of a leaky condenser in series with an inductive resistance, the oscillations being damped. The amplitude of the oscillations is proportional to the real part of e^{Pt}, where $P=p+j\alpha$ and $p=2\pi n$, n being the frequency and α the damping coefficient. Hence the volt drop down the resistance is $R+jPL$, and that down the condenser is $(S+jPC)^{-1}$. The sum of these two is zero, which leads at once to the equation

$$P=p+j\alpha=j\left(\frac{R}{2L}+\frac{S}{2C}\right)\pm\sqrt{\frac{1}{CL}-\left(\frac{R}{2L}-\frac{S}{2C}\right)^2},$$

giving the frequency and damping. The formulæ can be confirmed by oscillograms taken at low frequency with a Duddell oscillograph,

and a number of these were shown demonstrating the accordance of fact with deductions from the formula. In the same manner the case of the coupled circuits was considered and the E.M.F. equations written in the form $Tim_1I_1+jMP_2I_2=0$, $jMP_1I_1+Tim_2I_2=0$. When Tim stands for $(R+jPL)+(S+jPC)^{-1}$. Eliminating the currents I_1 and I_2, we have an equation which can be solved for p. Taking the reduced case of non-leaky condensers, tuned circuits, and zero resistance in the secondary, Dr. Fleming deduced the equation—

$$\{p^2CL(1-k)-1\}\{p^2CL(1+k)-1\}-a_1^2CL(1-(1+k^2)p^2CL)=0,$$

which shows that there are in general, oscillations of three frequencies in the circuits. This was confirmed by photographs of oscillograms and diagrams of resonance curves.

DISCUSSION.

Prof. S. P. Thompson admired Dr. Fleming's ingenious method, which was, he considered, much more intelligible to students than writing down the differential equation and then integrating it. He also admired the beautiful oscillograms, and hoped it was possible to procure copies of them, as he considered every teacher ought to have a set.

Dr. W. H. Eccles also expressed the opinion that Dr. Fleming's method of solution would be valuable to teachers. He also pointed out that the leakage S of the condenser could be so adjusted as to neutralise the effect of the external resistance R on the period, and suggested that this fact might be made use of to measure high frequency resistances.

Prof. G. W. O. Howe stated that he had used a similar method in teaching. It was very important to clearly bring before students what was happening in the various circuits. The oscillograms could be used to determine the high-frequency resistance, as the log. dec. of the train of oscillations could be measured very accurately.

The Author, in reply, stated that he possessed the original copies of the oscillograms, and would be pleased to supply copies to teachers desiring them.

XXIII. *An Exhibition of Braun Cathode-Ray Tubes and an Electrostatic Machine for Working them, used as a High-Frequency Oscillograph.* By Dr. J. A. Fleming, *F.R.S.*

Received April 15, 1913. Read March 14, 1913.

For the delineation of extra high-frequency electrical oscillations no mechanical oscillograph is available, and the only possible means is by the aid of a Braun cathode ray tube and phosphorescent screen.

The tubes used by the writer have been made by Messrs. Müller-Uri, of Brunswick, Germany, and have an extra large bulb holding a Willemite screen about 10 cms. in diameter. The tubes are about a metre in length. They have a special cup-shaped cathode and also a vacuum reducing device for letting down the vacuum if the tube becomes too hard. Owing to the sensitiveness of the cathode ray to magnetic fields and to the consequent displacement of the spot of light by the earth's field, it is well to surround the tube between the two perforated screens with a circular coil of wire through which a direct-current is passed. This creates a longitudinal magnetic field, and by giving this coil slight displacements the position of the cathode spot can be adjusted. The tubes contain also electrostatic field plates for producing electric displacement. To delineate a condenser discharge we have to make the cathode ray move with uniform, or nearly uniform, velocity across the screen horizontally and to move vertically with a displacement proportional to the discharge current of the condenser. We must also make the spot repeat its path exactly and rapidly. This is secured as follows: A small motor-driven alternator has a commutator put upon the shaft, having insulated sectors, and three brushes pressing on it as described in the previous Paper for charging at a battery and discharging a condenser through an inductive resistance. The electrostatic deflection plates of the Braun cathode tube are connected directly to the terminals of the alternator, or else to those of an interposed air-core transformer. The cathode ray is thus caused to sweep across the screen, and the amplitude of this motion is made much larger than the width of the screen, so that during the actual time of passage of the spot across the screen its motion is sensibly uniform.

The inductive discharge circuit of the condenser is made to be a coil which is so placed near the cathode tube that it deflects

the ray at right angles to the motion due to the electrostatic deflection. The commutator secures that this magnetic deflection will always take place just at the same instant when the cathode spot is passing across the screen. If these arrangements are properly made the cathode spot traverses the same path again and again, and the oscillogram is therefore visible as a permanent bright line.

To operate the cathode tube we need therefore some perfectly steady source of high potential. An ideal source is a storage battery of 10,000 or 20,000 small cells. This, however, is difficult to obtain and expensive. The next best appliance is a suitable electrostatic influence machine. The one used by the author, here exhibited, was made by Müller-Uri, of Brunswick, Germany, and is a two-plate influence machine of the Wimshurst type. The plates are of ebonite, and one is fixed and the other revolving. The ebonite plates have the metal sectors embedded in them, or rather sandwiched, between two plates, and communication with them established through small metal buttons or pins which appear on the surface. The revolving plate is driven by a small electric motor. This machine gives a very steady potential. It will give an 8 in. spark in air when in good order and will supply a current sufficient to decompose acidulated water. The makers claim that it will produce a current of 300 to 350 microamperes. It easily deflects a suitable galvanometer and can therefore be used to demonstrate the current-giving power of an electrostatic influence machine. The ebonite plates are kept in good order if rubbed occasionally with a mixture of powdered French chalk and absolute alcohol. They should, of course, be kept covered and not exposed to light.

This machine gives a very steady cathode spot on the phosphorescent screen, and, in conjunction with the arrangements above described, affords a means of visibly representing oscillations of much higher frequency than is possible with any mechanical oscillograph. At the present time there is a great necessity for some form of fairly portable oscillograph, which shall enable us to see the trains of oscillations in a radio-telegraphic transmitter and make an ocular inspection of the decrement possible. The only chance of doing this is by some form of cathode ray tube operated by an electrostatic machine or multicellular battery as above described, but at present the whole collection of apparatus required for this purpose is somewhat elaborate and not very portable.

As far as the experiments have gone, however, they seem to promise a means of doing it with certain appliances; and, further, work on the same subjects is being continued in our radiotelegraphic laboratory in University College.

ABSTRACT.

Some Braun cathode ray tubes used as high-frequency oscillographs, and also an electrostatic influence machine, giving a steady current of 300 to 350 microamperes for working them, were exhibited and described. The Braun tubes have electrostatic deflection plates in them and an embracing field coil for providing a longitudinal field to keep the cathode spot in a central position on the screen. The tubes are worked by connecting the deflection plates to the terminals of an alternator, whereby the cathode spot is made to move to and fro over the screen with a nearly uniform velocity. The shaft of the alternator carries a commutator by means of which a condenser is repeatedly discharged through an inductive coil which acts as a deflecting magnet on the oscillograph tube and displaces the cathode ray at right angles to the direction of the electrostatic displacement. In this way the spot has two motions, a uniform horizontal motion and a vertical oscillatory motion, the two being kept in step with each other. Hence a visual representation is given on the phosphorescent screen of the nature of the oscillation. The electrica machine above mentioned, made by Messrs. Müller-Uri, of Braunschweig, was shown in operation, and the current given by it was seen to be sufficient to cause decomposition of acidulated water in a lantern voltameter.

DISCUSSION.

Mr. A. Campbell stated that one German instrument maker had recently described in a Paper a method of improving the surface of ebonite. The ebonite was painted with a composition termed Bakelite and then heated to nearly the softening point of the ebonite, when the composition spread over the surface and produced a surface like amber, which was unaffected by either sunlight or moisture.

Prof. J. T. Morris stated that he had employed Braun tubes 10 years ago to find the maximum value of an alternating current, but found many difficulties in their use. He would like to know what was the actual voltage on the tube, and also whether the oscillograms were in a continuous line or were made up of a row of dots showing the cathode rays to be produced intermittently, as he believed was often the case with a Wimshurst machine. His brother, Mr. D. K. Morris, had suggested, several years ago, using a different number of sectors on the two plates in order to overcome this. He would like to know whether anyone had tried it. He asked if a high-potential battery would not give a more satisfactory result than the Wimshurst.

Dr. R. S. Willows asked if Dr. Fleming had used a hot lime cathode, as then the tubes could be run from 200-volt mains, and, as the cathode rays so produced had a much smaller velocity, much lower voltages could be used in the condenser discharges.

The Author stated that batteries would be ideal, but their price was prohibitive. Ordinarily Wimshurst machines do give a spotted appearance to oscillograms taken with Braun tubes, but the present machine gave a perfectly uniform and continuous curve. He had never tried hot lime cathodes.

XXIV. *Note on a Method of Observing the Flame Spectra of Halogen Salts.* By E. N. da C. Andrade, *B.Sc., Ph.D.*, 1851 *Exhibition Scholar of the University of London.*

Received March 31, 1913. Read March 14, 1913.

§ 1. It has been shown by Smithells* that if chlorine be introduced into a flame containing the luminous vapours of certain metals—*e.g.*, lithium or strontium—it destroys the coloration. To demonstrate this it is best to introduce the salt by means of a Gouy sprayer ; the effect can then be shown by passing the air supplied to the burner over chloroform. The effect cannot be shown by introducing a bead of the salt into the chlorinated flame, because the great volatility of the chloride formed causes the metal to vaporise so much faster that the chlorine is no longer present in sufficient excess to ensure the greater part of the vaporised metal being chemically combined. If the salt be sprayed, however, the amount of metallic vapour present is not affected by the introduction of the chlorine, which then completely extinguishes the colour.

§ 2. When introducing a bead of salt into a chlorinated flame on a nickel wire, I noticed that the nickel wire coloured the flame independently of the presence of any salt. This led me to introduce wires of other metals into the chlorinated flame, in the hope, afterwards justified, of obtaining characteristic spectra. The effect is very striking with copper. An ordinary copper wire in the chlorinated flame gives an intense blue in the lower part of the flame and a strong red in the upper part ; the spectrum turns out to be the same as that obtained in another way for copper chloride by Smithells,† and investigated in detail by him. The method described allows the different emissions in the different zones of the flame to be shown remarkably clearly.

§ 3. I divide the flame into four parts : the edge E, the outer mantle M_1, the inner mantle M_2, and the cone C (Fig. 1). Copper gives in the edge a red coloration, in the outer mantle blue ; the inner mantle is in general uncoloured. If the wire be held in the cone, however, a streak of blue vapour rises from it

* Smithells, Dawson and Wilson, "Phil. Trans.," 193A, 1900, p. 121. There is here a reference to a previous Paper by Smithells which contains a brief remark on the point in question.

† Phil. Mag." (V.), 39, p. 122, 1895.

into the inner mantel. Holding the wire through the flame just touching the cone, all these colorations can be seen at once in their respective zones, the division being very clearly marked.

The other chloride spectra which I have so far observed are those of nickel, cobalt and iron, the flame spectra of which do not seem to have been fully observed before in the cases where they have been observed at all. Introducing a nickel wire into the chlorinated flame I find in the edge a green coloration, in the outer mantle lavender, from the cone pink.*

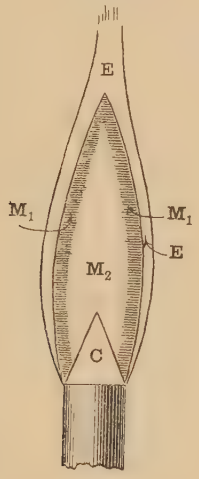


FIG. 1.

Not having cobalt wire, I introduced the glowed salt, which gives no colour in the ordinary flame, on a platinum wire. The flame is then red in the edge and a bluish pink in the outer mantle. Iron gives a yellow coloration, not due to traces of sodium.

§ 4. By heating the wires electrically while they are in the flame it can be shown that the emissions are probably not temperature effects. For instance, the wires may be heated electrically to melting in the edge of the flame without changing the character of the emission ; its intensity increases somewhat with rise of temperature, however.

* P. J. Hartog (British Assn. "Report," 1901, p. 613) makes mention of a red coloration from the cone, and a *temporary* purple which flashes out and disappears.

§ 5. The chloride spectra obtained in the manner described have certain general characteristics. In all cases there is a continuous background; further, the spectra are banded spectra. Copper chloride, as is well known, shows brilliant bands in the blue and green with a faint continuous background; the green bands are due to the oxide, according to Smithells (*loc. cit.*). The red coloration in the edge gives a continuous spectrum. With nickel the green in the edge is continuous; we have for the spectra in other parts of the flame a continuous background, with bands in the red, blue-green and violet. The red bands are strong in the cone and cause the pink tint. With cobalt there is a red continuous spectrum in the vapour in the edge of the flame, a continuous spectrum with bands in the green and blue superposed on it for other parts of the flame. With iron, apparently in any part of the flame, we have a continuous background with bands in the red, green and yellow, the strongest being the yellow band. It seems that for the different metals the continuous background has its maximum intensity in different places characteristic of the particular metal, as Lenard* has shown to be the case for the continuous backgrounds of the alkali metals. The fact that all these spectra are banded accords with the view that banded spectra may be attributable to molecules, line spectra to atoms.

Attempts have been made to get chloride spectra of tin, lead and mercury by bringing the vapours of these metals into the flame. Characteristic colorations are produced, but at present have not been resolved spectroscopically.

§ 6. I have made some experiments with flames into which the vapours of bromine and iodine respectively have been introduced. A very small amount of bromine vapour is sufficient to extinguish completely the strong colour produced by sprayed solutions of lithium salts. Iodine vapour produces the same effect. Wires of some of the metals already mentioned give characteristic spectra in these flames; these await further examination.

§ 7. A few observations have been made on the deviation in an electric field of the luminous streak of chloride vapour. Lenard† showed in 1902 that the luminous vapour from a bead of salt in a flame is deviated in an electric field, the amount of deviation depending on the metal (and, of course, on the

* "Annalen der Physik" (IV.), 17, p. 208, 1905.
† "Annalen der Physik" (IV.), 9, p. 642.

upward velocity of the flame gases, and the strength of the field). G. Ebert in 1911 showed that in the case of strontium this deviation decreased when the vapour was decolorised by chlorine in the flame, tending to become zero as the intensity of coloration decreased. Lenard remarked that copper chloride is deviated in the electric field; he apparently used a bead of the salt, and does not mention the different colorations. From present observation it seems that the red colour is deviated partly to the positive and partly to the negative electrode. The blue goes mainly to the negative electrode. The deviation is, however, especially marked in the blue vapour from the cone, the vapour behaving as if strongly positively charged. Similar results are found for nickel. There is no doubt that the luminous chloride vapour is strongly charged, the carriers (which emit band spectra) being mainly positive, but negative also existing.* Very rough estimates of the velocity of migration of the carriers show that the carriers probably alternate the positive state with the neutral, as assumed by Lenard for the carriers of the line spectra.† As we do not yet know in which state the carriers actually emit the light, we cannot, however, confirm or contradict Stark's assumption that the carriers of the band spectra are neutral.‡

§ 8. This preliminary note calls attention to the two classes of metallic chlorides :—

1. Those which are non-luminous in the flame; this class apparently consisting of the alkaline and alkaline earth metals.

2. Those which give characteristic banded spectra with continuous grounds.

A new modification of method has been described for obtaining halogen compound spectra of some metals, which enables us to examine with ease the emission of the different flame zones, and has led in some cases to spectra apparently new. The electrical behaviour can also be easily examined by this method. It has been pointed out that, while the vapours of non-luminous chlorides are unchanged, the luminous chloride vapours are strongly charged. I hope to study in more detail the structure of the spectra of the different halogen salts, and the electrical behaviour of the vapours, with a view to getting information on the mechanism of the emission of banded spectra.

* *Cf.* E. N. da C. Andrade, "Phil. Mag.," July, 1912, p. 16.
† *Loc. cit.*, 1902.
‡ *See* "Atomdynamik," Vol. II., p. 138.

ABSTRACT.

If a flame containing a large amount of chlorine be prepared by passing the air supplied to a colourless gas flame over chloroform, wires of certain metals—copper, nickel, iron, for instance—held in the flame give characteristic colorations in the different zones of the flame. These are due to the chlorides of the metal, which can exist undissociated, in some zones of the flame at least, in the presence of excess chlorine. The chloride spectrum of copper is well known (of Smithells) but the chloride spectra of nickel, cobalt, and iron chloride do not seem to have been fully observed before. The method makes it easy to observe the different emissions which take place in the different zones, and also the electrical migration of the vapour discovered by Lenard in 1902. All the chloride spectra have certain common characteristics. Attention is called to the fact, discovered by Smithells, that for the vapours of some metals—*e.g.*, lithium, strontium—the coloration produced in the flame is destroyed by chlorine. In such cases the vapours are not electrically charged, while in the case of the metallic chlorides which give characteristic spectra in the flame the vapours are strongly charged.

By bringing wires into flames containing bromine and iodine compound spectra were observed in some cases.

DISCUSSION.

Mr. A. CAMPBELL asked if Dr. Andrade could give an explanation of the blue coloured flame obtained when NaCl is thrown on to a fire.

The AUTHOR replied that the flame was the spectrum of copper chloride, there being a sufficient impurity of copper in the coal.

XXV. *On the Stretching and Breaking of Sodium and Potassium.* *By* Bevan B. Baker, *B.Sc., University College, London.*

Received April 1, 1913. Read March 14, 1913.

The nature of the structure of solid bodies, and in particular of metals, is a subject which, in spite of its importance, is rather obscure and difficult to determine. The present investigation will, it is hoped, throw a little light on some aspects of this elusive question.

Preliminary Investigation.—Some metals, when in the form of wires, on being stretched break off suddenly at one point, whereas others, plastic metals, thin down and finally collapse to a point at the place where they break.

Some months ago I had occasion to experiment on the behaviour of wires of metallic sodium when in a state of tension. The wires were formed by squeezing the metal through a round hole of about 2 mm. diameter into a bath of pure paraffin oil, to preserve the wire from oxidation. The wire, so prepared, was found not to collapse to a point nor to break off suddenly on stretching, but to collapse from two opposite sides only, into a chisel end.

To test whether this effect was due at all to the method by which the wires were constructed it was decided to form the wires by melting the sodium and allowing it to solidify in a glass tube.

Method of Forming the Wire.—A glass tube, whose internal diameter was that of the wire required, was sealed on to a much wider tube at one end, and at the other was connected by rubber tubing with an ordinary glass tap (*see* Fig. 1). The tube so formed was filled almost entirely with pure paraffin oil, which had been carefully dried, and the tap was closed. Lumps of metallic sodium were scraped free from oxide under oil and were introduced into the wider part of the tube. The tube was then placed inside a cylindrical electric furnace arranged in a vertical position, so that almost all of the narrow tubing and about half of the wider tubing was inside the furnace. The furnace was heated to about 180°C. and was kept at that temperature for a short time. The tap was then opened slightly to allow the oil to run slowly

out of the tube; at the same time the now molten sodium ran down into the narrow tube. When sufficient oil had run out the tap was closed and the whole apparatus was allowed to cool. When cold the tube was removed from the furnace and the tap and rubber tubing were taken off. By introducing a rod, which just fitted the inside of the narrow glass tubing, the mass of solid sodium was easily pushed out into a large bath of paraffin oil.

In this way it was found possible, in a simple manner, to

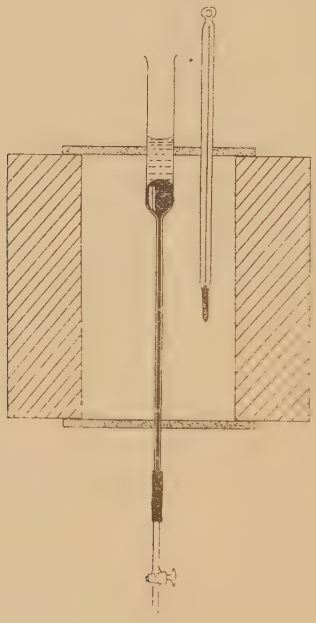


Fig. 1.

form wires of considerable length and having a remarkably smooth surface.

Description of the Phenomenon.—When the wire prepared in this way was stretched, it was found not to break off almost directly in one place, as the wire made by squeezing had done, but to thin, for some time, quite uniformly all along its length. This thinning was not, however, a collapse from all sides, but exhibited the same phenomenon of collapsing only from two opposite sides that the other wire had shown. For

further extension the wire gave way in certain places and broke in the chisel form.

Moreover, for wires prepared in this way a still further phenomenon became apparent which was scarcely observable for wires made by squeezing. As the wire was extended there began to be apparent on the surface markings in the form of two sets of equidistant rings encircling the wire, crossing one another and lying in planes sloped at an angle of about 45 deg. to the axis of the wire. Members of the two sets touched along the line of greatest thinning, and bisected each other along the line where no thinning took place (*see* plate 1). As the wire was still further stretched up to the breaking point the rings remained unchanged in position or slope, the only difference being that they became more clearly marked. It is noteworthy that usually one set of rings was much more distinct than the other. Microscopic examination made it clear that the rings were actual ridges on the surface.

Some experiments were also made in exactly the same way with metallic potassium, and it was found to contract and form rings precisely like sodium ; there was no apparent difference in their behaviour.

Dr. Andrade had independently noticed that wires of solid mercury when stretched contract to a chisel point. On observing the rings formed on sodium wires he searched for similar rings on mercury, and found that they were formed, only that they were very much finer and closer together than in the case of sodium and potassium.

Explanation of the Effect.—The phenomenon thus appears to show an asymmetric structure for the metals sodium, potassium and mercury. They appear to exhibit the behaviour of plastic metals in one direction and of brittle metals in the direction at right angles.

The effect suggests that the effective portions of the metal may be in the form of cubes set so that two opposite edges are horizontal and in the same vertical plane. When vertical extension of the wire takes place, and there arises a corresponding tendency for lateral contraction, it is clear that the faces inclined at 45 deg. may slip over the similar faces of adjacent crystals, and thus allow of a rearrangement in a narrower space ; on the other hand, the vertical faces can only slip over similar vertical faces, and clearly no contraction in the direction at right angles to such faces can take place.

It is clear that the crystals would pack as closely together

as possible, and would form in layers which would lie so as to make with the axis of the wire an angle of 45 deg. The rings observed on the wire would then be due to the edges of the crystals protruding at the surface of the wire

Description of Plate.

(*a*) Wire of sodium of 3 mm. diameter drawn out to breaking. View of specimen in the plane in which no contraction takes place.

(*b*) Same specimen as (*a*). View of specimen in the plane in which maximum contraction takes place.

ABSTRACT.

The author described how wires made of metallic sodium and potassium collapse when stretched, not to a point, as is the case with most plastic substances, but from two opposite sides only, into a chisel end.

The wires upon which experiments were conducted were made in two ways—firstly, by pressing the metal through a small hole into a bath of paraffin oil to hinder oxidation; and, secondly, by running the metal, molten under oil, into a glass tube and allowing it to solidify. Wires made by both the above methods showed the same behaviour on stretching.

Wires made by the second process also showed, on extension, two sets of equidistant rings on their surface, each inclined at an angle of 45 deg. to the axis, the rings of opposite sets touching along the line of greatest thinning and bisecting one another along the line at which no thinning takes place.

Dr. Andrade has also noticed the same effects of breaking and forming rings with wires made of solid mercury.

The author suggested an explanation of the phenomenon, based on the assumption that the portions of the metal brought into play are in the form of cubes. Such cubes when placed so that a plane through two opposite edges was parallel to the axis of the wire would allow of lateral contraction by faces sliding over one another in one direction only and not in the direction at right angles.

DISCUSSION.

Dr. L. N. G. Filon asked if the section of the stretched wire was accurately elliptical. It was interesting to note that the angle of the rings did not change when the wire was drawn out.

Prof. A. W. Porter remarked that it looked as if the metal possessed a lattice-like structure.

The Author stated that he had not measured the sections to see if they were accurately elliptical.

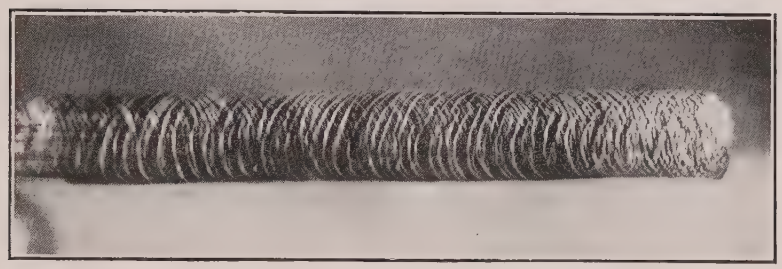


FIG. (*a*).

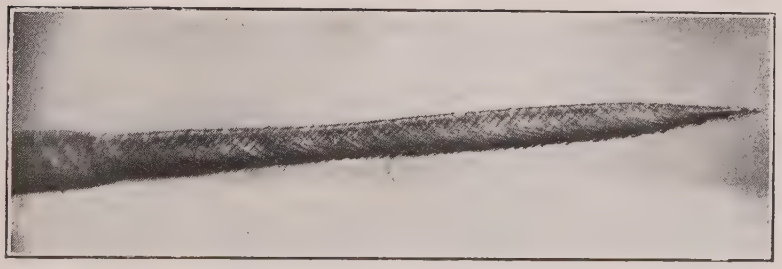


FIG. (*b*).

To face p. 238.]

XXVI. *Note on Optical Imagery.** *By* T. Smith, *B.A., National Physical Laboratory.*

Mr. T. Smith (communicated remarks) stated that he had been interested in looking through Mr. Bower's Paper, but he did not entirely agree that the treatment suggested therein was the best one to adopt with beginners. By starting with the velocity ratio definition of μ it was true that a physical conception of its meaning was obtained, but there was the disadvantage that a pupil could not determine it for himself with the aid of the simple apparatus that would be at his disposal. The better way appeared to be the older one of defining the refractive index as the ratio of sines after the pupil had found experimentally that this ratio was constant; he should be able to find for himself all the laws of refraction. Many assumptions were made in the velocity ratio definition which would not be realised by the student. For instance, he would not easily understand the necessity for an experimental basis for the law relating to the multiplication of refractive indices—$\mu_{A-C}=\mu_{A-B}\times\mu_{B-C}$. The statement of laws in such a way as to veil the assumption of facts which could easily be verified experimentally was to be deprecated. Moreover, when the velocity ratio conception was used the more obvious method to follow with a view to enabling the student to visualise what was taking place was to draw successive wave fronts, since it was easy to show that the effect on a wave front of light of encountering a refracting or reflecting surface of the dimensions with which Geometrical Optics dealt was to alter its curvature by a constant amount when the incidence was normal or nearly normal.

For tracing through a lens system rays to which the conventions of geometrical optics did not apply, it seemed to him best to use Young's construction to determine the directions of the rays, and to calculate in turn for each surface the positions of the primary and secondary focal lines from the algebraic formulæ. These formulæ were most simply derived from the algebraic conditions that the length of the optical path between the object and the so-called image for the chief ray should be equal to that for a slightly displaced ray: each of the formulæ in question could be proved in a couple of lines in this way, and would be more easily followed by a student than the geometrical proofs.

In paragraph 18 (vii.) the author used the term "image" as applied to the point I. In such a position as was shown in Fig. 6 "image" could not have any definite meaning until a special one had been assigned to it: the ordinary definition of geometrical optics did not apply in this case. In the absence of a special definition it appeared desirable to avoid using the word in the presence of aberrations.

He was not inclined to agree with the author in allowing more than was necessary of a pupil's attention to be given to what happened to a ray in its passage between the first and final media of a refracting system. The great advantage of the Gaussian method of treating lens systems was that all systems with a finite focal length might be investigated at one and the same time, and the whole subject could be fully treated in a few pages. The present-day text-books made the subject much more lengthy and difficult than was necessary by considering first the case of a simple (thin) lens, to which, though of little practical importance, the attention of pupils was chiefly directed. Then followed thick lenses and afterwards more complex cases. As a result of dividing the subject up into so many different sections, which

* A contribution to the discussion on Mr. W. R. Bower's Paper on pp. 160-177 of the present volume.

were treated independently, the student usually failed to grasp any part but the thin lens, and in examinations avoided simple questions on thick lenses. It appeared a wiser course to master at first the very small amount of algebra necessary to treat the general case at the beginning of the subject, and to scrap the special formulæ dealing with individual cases which were of no practical importance. This would help to get rid of much misconception on geometrical optics. For instance, it would help the student to realise that the focal length of a lens was not the distance to the principal focus from the external surface of the lens, and that, because the thickness of the glass of a simple lens was quite small compared with the focal length, the formulæ for a "thin" lens did not necessarily give a fairly accurate value for the focal length. Consider, for instance, a lens of refractive index 1·5 placed in air of thickness 1 cm. and radii of curvature 100 cm. and 99 cm., the centres of curvature being coincident. The focal length is 29,700 cm., which is quite large compared with the thickness 1 cm. Yet the principal points are not near the surfaces but at the centre of curvature. The formula for thin lenses would make the focal length 19,800 cm. It should be noted that this was not at all an extreme case, but that the curves were very flat, the thickness being small compared with either radius of curvature, and in fact the flatter the curves the worse the approximation given, assuming that the properties of thin lenses held. It was instructive to slightly alter the curvature of one of the surfaces to make the lens one of long positive focal length and determine the distances of the principal foci from the surfaces.

For any other purposes than those already mentioned, viz., to give a visual impression of what was taking place at a refracting or reflecting surface, and for graphically tracing rays through a system, the algebraic methods seemed to him much simpler than any geometrical methods were likely to be. It would be easy to put these into a form in which the positions of all the cardinal points with respect to the extreme surfaces and the focal lengths could be calculated for a combination of quite a large number of surfaces in a few minutes. He had ventured to put down below the forms which he had found most convenient. It was assumed that there were n surfaces identified by being numbered from 1 to n: the refractive indices are $\mu_0, \mu_1 \ldots \mu_n$: let $\kappa_p = \frac{\mu_p - \mu_{p-1}}{r_p}$ = the power of the pth surface: $K_{a,b}$ = the power of the system of surfaces bounded by the ath and bth inclusive: t_p is the axial thickness between the surfaces bounding the medium μ_p.

$$K_{a,b} = 0 \text{ if } a > b \quad \text{and} \quad K_{a,a} = \kappa_a.$$

Then

$$\left.\begin{aligned} K_{1,p} &= K_{1,p-1} + \kappa_p \frac{\partial K_{1,p}}{\partial \kappa_p} \\ \frac{\partial K_{1,p}}{\partial \kappa_p} &= \frac{\partial K_{1,p-1}}{\partial \kappa_{p-1}} - \frac{t_{p-1}}{\mu_{p-1}} K_{1,p-1} \end{aligned}\right\} \quad \ldots\ldots\ldots (1)$$

It would easily be seen that by using the formulæ

$$\left.\begin{aligned} K_{p,n} &= K_{p+1,n} + \kappa_p \frac{K_{p,n}}{\kappa} \\ \frac{K_{p,n}}{\partial \kappa_p} &= \frac{\partial K_{p+1,n}}{\partial \kappa_{p+1}} - \frac{t_p}{\mu_p} K_{p+1,n} \end{aligned}\right\} \quad \ldots\ldots\ldots (2)$$

the values of $K_{1,n}$ given by (1) and (2) would be the same, and that

$$\frac{\partial K_{1,n}}{\partial \kappa_1} \cdot \frac{\partial K_{1,n}}{\partial \kappa_n} - K_{1,n} \frac{\partial^2 K_{1,n}}{\partial \kappa_1 \partial \kappa_n} = 1. \quad \ldots\ldots\ldots (3)$$

Let (x_0, y_0) be the co-ordinates of a point on an incident ray inclined at θ_0 to

the axis, x_0 being measured along the axis from the vertex of the last surface, and y_0 perpendicular to the axis. Similarly let $(x_n y_n)$ be the co-ordinates of a point on the refracted ray inclined at θ_n to the axis, x_n being measured from the vertex of the last surface and y_n perpendicular to the axis. The law of refraction would at once give

$$\mu_n\theta_n=\mu_0\,\theta_0\frac{\partial \mathrm{K}_{1,n}}{\partial\kappa_1}+(x_0\theta_0-y_0)\mathrm{K}_{1,n} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and

$$x_n\theta_n-y_n=\mu_0\theta_0\frac{\partial^2 \mathrm{K}_{1,n}}{\partial\kappa_1\partial\kappa_n}+(x_0\theta_0-y_0)\frac{\partial \mathrm{K}_{1,n}}{\partial\kappa_n}. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The relation (3) would at once give from (4) and (5)

$$u_0\theta_0=\mu_n\theta_n\frac{\partial \mathrm{K}_{1,n}}{\partial\kappa_n}-(x_n\theta_n-y_n)\mathrm{K}_{1,n} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and

$$x_0\theta_0-y_0 \quad .\quad -\mu_n\theta_n\frac{\partial^2 \mathrm{K}_{1,n}}{\partial\kappa_1\partial\kappa_n}+(x_n\theta_n-y_n)\frac{\partial \mathrm{K}_{1,n}}{\partial\kappa_1}, \quad . \quad . \quad . \quad . \quad (7)$$

results which could be written down from symmetry with (4) and (5).

Equations (4) and (6) contain the whole theory of co-axial lens systems, however complicated, which have a finite focal length. For instance, considering points on the axis, it will be seen that all rays starting from the point $x_0=-\mu_0\dfrac{\dfrac{\partial \mathrm{K}_{1,n}}{\partial\kappa_1}}{\mathrm{K}_{1,n}}$ emerge parallel to the axis, so that this point is the first principal focus. Similarly the position of the second principal focus is given by

$$x_n=-\frac{\mu_n\dfrac{\partial \mathrm{K}_{1,n}}{\partial\kappa_n}}{\mathrm{K}_{1,n}}$$

Putting x_0 and x_n equal to these values, the equations (4) and (6) become

$$\mu_0\theta_0=-y_0\mathrm{K}_{1,n},$$

and

$$\mu_0\theta_0=y_n\mathrm{K}_{1,n},$$

showing that $\dfrac{\mu_0}{\mathrm{K}_{1,n}}$ and $\dfrac{\mu_n}{\mathrm{K}_{1,n}}$ measure the scale on which the image of an infinitely distant object would be formed. These quantities are called focal lengths. Again the points on the axis for which

$$x_0=\frac{\mu_n-\mu_0\dfrac{\partial \mathrm{K}_{1,n}}{\partial\kappa_1}}{\mathrm{K}_{1,n}},$$

and

$$x_n=\frac{\mu_n\dfrac{\partial \mathrm{K}_{1,n}}{\partial\kappa_1}-\mu_0}{\mathrm{K}_{1,n}}$$

have the property that any ray passing through one of them emerges in the same direction as that in which it is incident, and they are called nodal points.

When

$$x_0=\mu_0\frac{\dfrac{1}{m}-\dfrac{\partial \mathrm{K}_{1,n}}{\partial\kappa_1}}{\mathrm{K}_{1,n}} \quad \text{and} \quad x_n=\mu_n\frac{\dfrac{\partial \mathrm{K}_{1,n}}{\partial\kappa_n}-m}{\mathrm{K}_{1,n}} \quad . \quad . \quad . \quad . \quad (8)$$

we obtain in both cases for points on the axis

$$\mu_0\theta_0 = m\mu_n\theta_n, \quad \ldots \quad (9)$$

showing that these points are in the relation of object and image.

Putting in these values of x_0 and x_n when the points considered are off the axis we obtain

$$y_n K_{1,n} = \mu_0\theta_0 + m\mu_n\theta = m y_0 K_{1,n}, \quad \ldots \quad (10)$$

showing that for the object and image planes defined by (8) the linear magnification is m. Putting $m=1$, we get the equations for the unit planes

$$x_0 = \mu_0 \frac{1 - \dfrac{\partial K_{1,n}}{\partial \kappa_1}}{K_{1,n}}$$

and

$$x_n = \mu_n \frac{\dfrac{\partial K_{1,n}}{\partial \kappa_n} - 1}{K_{1,n}}.$$

It is evident that all the usual relations can be quickly obtained by change of origin or by other simple devices. For instance, from (9) and (10) we have $\mu_0\theta_0 y_0 = \mu_n\theta_n y_n$, a result of great practical importance which is too often neglected.

This method presents no difficulties when the fundamental algebra of (1) and (2) is mastered. It may be as well to illustrate how practical calculations of focal lengths, &c., are best arranged. The following constructional data are very approximately those given in a patent specification for a photographic lens; we may assume that the unit of length is 1 in. The radius of curvature is positive when the surface is convex to the incident light:—

Surface.	μ	r	t
	1		
1		+3	
	1·53		0·2
2		−30	
	1		0·8
3		−4	
	1·60		0·1
4		+4	
	1		0·9
5		∞	
	1·61		0·2
6		$-3\frac{1}{3}$	

Thus we have

$\kappa_1 = +0{\cdot}177$ $\quad t_{1/\mu_1} = 0{\cdot}131$
$\kappa_2 = +0{\cdot}018$ $\quad t_{2/\mu_2} = 0{\cdot}8$
$\kappa_3 = -0{\cdot}15$ $\quad t_{3/\mu_3} = 0{\cdot}063$
$\kappa_4 = -0{\cdot}15$ $\quad t_{1/\mu_4} = 0{\cdot}9$
$\kappa_5 = 0$ $\quad t_{5/\mu_5} = 0{\cdot}134$
$\kappa_6 = +0{\cdot}183$

Then from equations (1)

—	$\frac{\partial K_{1,p}}{\partial \kappa_p}$.	$\kappa_p \frac{\partial K_{1,p}}{\partial \kappa_p}$.	$K_{1,p}$.	$-\frac{t_p}{\mu_p} K_{1,p}$.
$p=1$	1	0·177	0·177	−0·023
$p=2$	0·977	0·017	0·194	−0·156
$p=3$	0·821	−0·123	0·071	−0·004
$p=4$	0·817	−0·122	−0·051	+0·046
$p=5$	0·863	0	−0·051	+0·007
$p=6$	0·870	0·159	0·108	...

Similarly, by working through the system from the other end we have from equation (2) :—

—	$\frac{\partial K_{p,6}}{\partial \kappa_p}$.	$\kappa_p \frac{\partial K_{p,6}}{\partial K_p}$.	$K_{p,6}$.	$-\frac{t_{p-1}}{\mu_{p-1}} K_{p,6}$.
$p=6$	1	0·183	0·183	−0·024
$p=5$	0·976	0	0·183	−0·165
$p=4$	0·811	−0·122	0·061	−0·004
$p=3$	0·807	−0·121	−0·060	+0·048
$p=2$	0·855	+0·016	−0·044	+0·006
$p=1$	0·861	+0·152	0·108	...

Thus, the focal length is $\frac{1}{0\cdot108}$ = 9·26 in., and the distances of the principal foci from the first and last surfaces are $\frac{0\cdot861}{0\cdot108}$ = 7·97 in. and $\frac{0\cdot870}{0\cdot108}$ = 8·05 in. respectively. Every figure is recorded above that is needed by a pupil using a slide rule. The whole calculation takes only a few minutes and the final agreement provides a good check on the accuracy of the figures. The case of telescopic systems could be dealt with equally simply.

In conclusion, he would refer to the use of the term "principal plane." He had been under the impression that the word principal was restricted to the cases when the first and last media were the same, so that the unit points and nodal points coincided. This, whether generally adopted or not at present, appeared to be a convenient usage, and it would be advantageous in the more general case to use the descriptive term "unit" plane rather than the more conventional term "principal."

Mr. W. R. Bower replies as follows :—The critics of the Paper on "Optical Imagery" have, as requested, paid attention to the value of the method to the student, but misconception has arisen concerning the meaning of "beginners." When speaking of these the author had in mind not so much those who are learning the alphabet of optical science, but somewhat more advanced pupils, as for instance, the students engaged in a higher course of physics who would take mathematics as a subsidiary subject. Such students would have proved the empirical rules of refraction at an earlier stage, and to these, perhaps, no harm is done by using, as a basis, the definition of μ as a velocity-ratio. However, teachers who object to this, could start by reversing § 3 (i). In the Paper as it stands there is no use for the sine law of refraction. Judging by the criticisms offered, it now seems important to say that the Paper and diagrams as published were not intended for students. The extent of the Paper would have been unduly increased if it had been arranged expressly for them. It is anticipated that a teacher will himself divide the diagrams and expand the text as he may think necessary. The

student would then make careful drawings, by using mathematical instruments, of the cases considered. These would form an approximate sequel to laboratory experiment and a substitution for some of the problems of the exercise class. With many students undoubtedly an intelligible graphic method leads to a better understanding of a subject than that obtained by algebraic methods only. And in optics especially the value of well executed drawings, giving a true picture of the course of the rays, is possibly as great as that of the mechanical models that are educationally of so much advantage in the various branches of science. The method suggested in the Paper should practically have the value of a moving model. The graphic method of the Paper develops in a natural manner, and the usual algebraic formulæ may be derived with very little special artifice. It should therefore be intelligible and educational. The opinions of eminent teachers on the advantage of models are stated in Dr. S. P. Thompson's Presidential Address in Vol. XVII. of the " Proceedings " of this Society.

The author agrees with Mr. Smith that our present text-books make the subject more difficult and of much greater length than is necessary. Something will have been gained by this discussion if the thick lens obtains in the future a warmer welcome than it has received in the past. Cornu's junction-point, too, has so far had very scanty recognition in this country.

XXVII. *The Spectroscopic Resolution of an Arbitrary Function.*

By C. V. BURTON, *D.Sc.*

RECEIVED MARCH 7, 1913. READ APRIL 25, 1913.

1. From time to time, since Dr. Schuster made known his conception of the periodogram, I have attempted to devise some quasi-spectroscopic method of analysing graphic records —not realising that the resolution could be affected by the spectroscope itself. Though I have not at present the means of working out the practical details of the method, a short communication on the subject may perhaps be justified by the simplicity and completeness of the optical theory, and the advantage which should result to more than one branch of science from a successful application of the principle proposed.

2. An ordinary grating has periodic rulings, and a spectrum obtained by means of it is characteristic of the radiation entering the spectroscope slit. On the other hand, if the radiation is homogeneous, while the distribution of the rulings is arbitrary, we obtain a spectrum characteristic of the *grating*. With reservations to be mentioned immediately, it should suffice if we can produce a transmission grating whose transparency (variable in one dimension only) is suitably related to the record in question. When this *equivalent grating* takes its place in a spectroscopic train, the slit being fed with homogeneous light, we shall be able to see or to photograph the spectrum of the record. The necessary reservations are: (i.) The effect of the transmission grating on any given element of the (normally) transmitted disturbance must be simply a reduction of amplitude, with no change of phase except such as is uniform over the whole extent of the grating,* and (ii.) the absorbing layer should be flat enough and thin enough to be sensibly equivalent to a plane surface whose power of absorption varies from point to point.

3. For convenience let the permeability (P) in the neighbourhood of any point be defined as the square root of the reciprocal of the "density," so that at normal incidence the

* This will be at least very approximately the case if the partial opacity is due to a distribution of small perfectly opaque granules, with transparent interspaces through which the luminous disturbance can penetrate.

proportion of luminous energy transmitted is P^2. A simple harmonic grating* is defined by

$$P=A_0+A_1 \cos px,$$

where A_0 and A_1 are constants of which A_0 is necessarily the greater, $2\pi/p$ is the grating interval and P is the permeability at every point of that vertical line whose horizontal co-ordinate is x. With the slit illuminated by homogeneous light of wave-length λ ($=2\pi/n$, say), and with the incidence normal, such a grating gives a central image of intensity proportional to A_0^2, and on each side a *single* spectral image, at an emergent angle θ given, in accordance with the ordinary grating formula, by

$$p=n \sin \theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

4. Assuming still the conditions postulated in § 2, consider the general case of a grating of width b extending from $x=-\frac{1}{2}b$ to $x=\frac{1}{2}b$. Let the permeability P be a function of x only, and equal to $P_0+\varphi(x)$, where P_0 is a positive constant, while the mean value of $\varphi(x)$ is zero. Within the limits $x=\mp\frac{1}{2}b$, the value of $P_0+\varphi(x)$ must evidently lie between 0 and 1 ; beyond those limits it vanishes. Let homogeneous light proceeding from a point source and rendered parallel by an object glass fall normally on the grating, and for simplicity suppose that this light is plane-polarised, the plane of polarisation being either parallel or perpendicular to the rulings. In either case the disturbance incident upon the grating may be specified by writing down the value of the electric vector as a function of the time. Let this be

$$A \cos nVt,$$

so that V is the velocity of light. The transmitted disturbance is accordingly

$$AP \cos nVt=AP_0 \cos nVt+A\varphi(x) \cos nVt.$$

5. The first term on the right gives rise to the central image and need not be further considered. To determine the effect of the remaining term, imagine a telescope of assigned focal length pointed at the centre of the grating in an azimuth inclined θ to the grating normal. The amplitude of the luminous disturbance reaching the focus of the telescope has to be expressed as a function of θ. Factors which are not variable with θ may be omitted ; nor are we here concerned with variations in a vertical sense, since these in practice would

* *Cf.* Rayleigh, "Wave Theory of Light," § 15.

be obliterated by substituting an illuminated slit for the ideal point source. Corresponding to the emergent angle θ, the disturbance proceeding from any point x of the grating suffers a relative retardation $x \sin \theta$, and, with an arbitrary change in the origin of time, the disturbance at the focus of the telescope is seen to be proportional to

$$\psi(\theta)\int_{-\infty}^{+\infty} \cos n(\mathrm{V}t - x \sin\theta)\varphi(x)\, dx, \quad . \quad . \quad . \quad (2)$$

in which the limits of integration might equally well have been given as $\mp\frac{1}{2}b$. The factor $\psi(\theta)$ in general differs from unity, because each element of the disturbance emerging in the plane of the grating is oblique to the direction of the beam considered. From symmetry $\psi(\theta)$ is an even function of θ, and in the ideal case, where the effect of the grating is entirely one of absorption, its form is simple: for an incident beam polarised perpendicularly to the rulings, $\psi(\theta)$ is constant and equal to unity; for polarisation parallel to the rulings, $\psi(\theta) = \cos\theta$. If the incident light is unpolarised (which implies that the homogeneousness is not mathematically absolute) the factor corresponding to $\psi(\theta)$ will be $\sqrt{\{\frac{1}{2}(1+\cos^2\theta)\}}$, the emergent beam being perceptibly polarised except for very moderate values of θ.

6. In virtue of (1) the expression (2) may be written

$$\left.\begin{aligned} &\psi(\theta)\{f_1(p)\cos n\mathrm{V}t + f_2(p)\sin n\mathrm{V}t\}, \\ \text{where}\quad &\begin{matrix} f_1(p) \\ f_2(p) \end{matrix} = \int_{-\infty}^{+\infty} \begin{matrix} \cos \\ \sin \end{matrix} pu\varphi(u)du \end{aligned}\right\} . \quad . \quad . \quad (3)$$

The corresponding intensity of illumination is

$$\mathrm{B}\{\psi(\theta)\}^2[\{f_1(p)\}^2 + \{f_2(p)\}^2], \quad . \quad . \quad . \quad . \quad (4)$$

where B is an instrumental constant. If the brightness has been mapped throughout the spectrum, and if the value of B and the form of the function $\psi(\theta)\}^2$ have been independently found, we shall be able to plot the values of $\{f_1(p)\}^2 + \{f_2(p)\}^2$ as ordinates against abscissæ proportional to p or to the grating interval $2\pi/p$.

7. Where only moderate values of θ are in question, so that θ^2 and higher powers may be neglected, $\psi(\theta)$ is sensibly a constant, and moreover the spectrum may be photographed on a flat plate. If the exposure law of the plate is known, the photograph may be regarded as equivalent to a periodogram,*

* *See* footnote to § 11.

the case being so far analogous to that of an ordinary spectrogram. In the present instance, however, the scale of abscissæ is not one of wave-lengths, but of *reciprocal* grating intervals (p).

8. Now by Fourier's theorem

$$\varphi(x)=\frac{1}{\pi}\int_0^\infty f_1(p)\cos pxdp+\frac{1}{\pi}\int_0^\infty f_2(p)\sin pxdp, \quad . \quad . \quad (5)$$

$f_1(p)$, $f_2(p)$ having the values defined in (3), and a full determination of the functions f_1, f_2 would constitute a complete Fourier analysis of $\varphi(x)$.

If we put

$$\left.\begin{aligned} &\frac{1}{\pi}[\{f_1(p)\}^2+\{f_2(p)\}^2]^{\frac{1}{2}}=\mathrm{K},\\ &f_1(p)/\pi\mathrm{K}=\cos\chi,\\ &f_2(p)/\pi\mathrm{K}=\sin\chi, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

(5) becomes

$$\varphi(x)=\int_0^\infty \mathrm{K}\cos(pt-\chi)dp, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and it is K^2 as a function of p which we are in a position to determine spectroscopically. We have yet to ascertain the phases of the various harmonic constituents, that is, to determine χ as a function of p. There are, broadly speaking, two cases in which the χ's have a definite significance. One is the case of a function $\varphi(x)$, whose effective values are confined to a limited range of value of the argument; χ is then in general a continuous function of p. The other is the case where $\varphi(x)$ is wholly or partly made up of absolutely pure harmonic constituents, corresponding to discrete values of p. The corresponding χ's are then precisely definite, and the greater the range over which our Fourier analysis is extended the more accurately they can be determined.

9. To find the χ's, but in each case with an ambiguity of half a period, we may proceed as follows: Imagine the grating bisected along a line perpendicular to the rulings, and one-half of it rotated through 180 deg. in its own plane, carrying with it its origin of x. There will then be a single vertical *line of agreement*, where like values of x come together. If now one-half of the grating is moved lengthwise micrometrically, any given line of the spectrum will show periodic variations of brightness, the minima being extinctions under proper conditions of adjustment. The minima will occur when the line of agreement is at a node of the harmonic grating constituent responsible for the line examined. The positions of the nodes having thus been determined, the ambiguity can in most cases be removed by integration over a relatively short portion of

the original graphic record. I think this plan might be quite useful in the case of a banded spectrum, derived from a function of limited range for, when the phase had been determined without ambiguity (say) for the dominant period of the band, the distribution of phase throughout the band could probably be followed without much difficulty. Perhaps the chief use of the spectroscopic method would be to indicate what periods are worth examining in more laborious ways.

10. Where the incident light is as nearly as possible homogeneous, and the periods of the grating are to be investigated, the spectroscopic conditions are in some respects unusual. The limited width of the grating still imposes a minimum interval which must exist between the centres of two lines if they are to be seen as separate ; but, given the grating as the subject of observation, no irrelevant broadening of lines is here implied. The grating is in fact analysed into simple harmonic gratings of infinite width, and, in the absence of aberrations, with a sufficiently narrow slit, the performance is only limited by the value of $\lambda/\Delta\lambda$, where $\Delta\lambda$ is the " effective range " of wave-length comprised in the incident radiation. Now $\lambda/\Delta\lambda$ can easily be 400,000 or more, so that we can count on all the spectroscopic detail to which our record entitles us, though for photographing it upon a convenient scale a rather long camera may be needed.

11. For moderate angles of emergence, as already stated, we have a sensibly uniform scale of *reciprocal* grating intervals, the middle of the scale corresponding with $2\pi/p=\infty$. The original graph will in many cases represent a function $\mathrm{H}\varphi(t)$ of the time t, H being in general a dimensional constant. In that case the spectrum seen will be disposed according to a sensibly uniform scale of *frequencies*.* The whole spectrum

* The function $\phi(x)$ within the range $x=\mp b/2$ has throughout been treated as the entire subject of analysis. This would be the natural mode of proceeding when, for example, $\mathrm{H}\phi(t)$ was the ordinate of a seismogram, whose outstanding features were confined to a correspondingly restricted space of time. $\{f_1(p)\}^2+\{f_2(p)\}^2$ is then characteristic of the function ϕ *as a whole*, and is a factor of what I propose to call the *integral periodogram*, a term which has Prof. Schuster's approval. On the other hand, as Schuster has pointed out, when the effective values of the function to be analysed extend with statistical uniformity over a wide range, of which only a relatively small *sample*, extending over a range b, is taken for analysis, the *average* value obtained for $\{f_1(p)\}^2+\{f_2(p)\}^2$ will be sensibly proportional to b, and on division by b will furnish quantities characteristic solely of the function in question. In all such cases $b^{-1}[\{f_1(p)\}^2+\{f_2(p)\}^2]$ is a factor of the ordinate of the periodogram, the term being now used in its accepted sense, without qualification. *Cf.* Schuster, " Proc. R.S.," A. 77, pp. 136–140 (1906).

being in light of one wave-length, the comparison of intensities, whether visually or photographically, is simplified. In particular the "density" in the neighbourhood of any point of the grating has a single definite value.

12. Remembering that it is the square root of the reciprocal of the photographic density whose values are to be (within a constant term) proportional to the ordinates of the record, we have first to determine the law connecting density with exposure, and in the apparatus which I have provisionally planned the problem of producing the equivalent grating from a given record then resolves itself into the suitable shaping of a cam. A good test of the performance would be to produce truly periodic gratings "equivalent" to pure sine curves of long and of short period, and to examine these in the spectroscope for harmonics. Various considerations indicate that a fairly open scale for the grating is desirable.

Note added May 20th, 1913.

The resolution of a transmission grating into harmonic constituents may be illustrated by a familiar example. Consider a grating consisting of a regular succession of narrow strips of greater and of lesser transparency alternately. The graph representing the distribution of permeability (§ 3) is in this case simply a "chess-board," and is readily analysable into a series of sine curves, each of which corresponds to an harmonic grating constituent. This compound character of the grating is evidenced by the existence of second and higher order spectra in addition to the spectrum of the first order. When the radiation examined by means of such a grating is sensibly homogeneous, a spectrum of any given order consists of a single fine line, and the aggregate of these lines on one side or the other of the central image is the spectrum of the grating.

Since the foregoing Paper was in type, I have had the advantage of discussing the matter with Prof. H. H. Turner, whom I have to thank for a number of valuable suggestions. As a result, the practical aspect of the method now appears more simple than I had thought possible, and I have begun making some preliminary tests.

Lord Rayleigh's Paper, referred to in his communicated remarks, seems to have been unaccountably overlooked by those interested in harmonic analysis. It was only a day or two before my own attention had been drawn to this Paper that I had realised the possibility of analysing a function by

means of a grating whose rulings are of variable *length* instead of variable intensity. The grating is thus reducible to a template, whose outline is the graph of the function to be analysed; but certain instrumental restrictions are at the same time imposed. The (vertical) slit of the spectroscope must be shortened to a pinhole, and attention must be confined to a linear horizontal strip of field, the central line of which is the horizontal axis of the diffraction pattern. These are precisely the conditions indicated and experimentally verified by Lord Rayleigh, who refers, in the Paper already cited, to the inconvenience which they entail. But as regards adaptability to practical needs, Lord Rayleigh's method seems to me distinctly more promising than that described above, for it demands no knowledge of the exposure law of a photographic plate, thus avoiding more than one serious difficulty, while the special drawbacks which have been referred to can be overcome by comparatively simple devices.

ABSTRACT.

An ordinary grating has periodic rulings, and a spectrum obtained by means of it is characteristic of the radiation entering the spectroscope-slit. But if the radiation is homogeneous, while the distribution of the rulings is arbitrary, we obtain a spectrum characteristic of the grating. It is thus found to be theoretically possible to resolve spectroscopically a given arbitrary function $\phi(x)$ into its harmonic constituents. The "permeability" of a photographic negative at any point being defined as the square root of the reciprocal of the "density," the first step is to make an "equivalent grating." This is a plate whose permeability (variable in one dimension only) has at any point x the value $A+B\phi(x)$, where A and B are constants. When this (transmission) grating takes its place in a spectroscope whose slit is fed with homogeneous light, the spectrum of the function $\phi(x)$ can be seen or photographed. Suitably interpreted, it gives us the periodogram of $\phi(x)$. A device is described which it is hoped may prove useful for determining the phases of the various harmonic constituents.

The theory of the proposed method of resolving functions is discussed, and is as complete as that of ordinary spectroscopy, while in one respect it is more simple; for, since the light entering the spectroscope-slit is entirely of one wave-length, the comparison of intensities of spectral lines (whether visually or photographically) is facilitated.

Some preliminary practical tests are now being made.

DISCUSSION.

Lord RAYLEIGH communicated the following remarks: In connection with Dr. Burton's interesting Paper I should like to draw attention to somewhat similar suggestions that I made in 1903 ("Phil. Mag.," V., p. 238). It is

to be wished that one or other of these optical methods should be applied to some practical problem of the analysis of an irregular curve.

The CHAIRMAN remarked that those who had gone through the labour of getting out a periodogram would appreciate any mechanical means of doing this. He understood that Dr. Burton considered Lord Rayleigh's method was better than his own.

The AUTHOR stated that he thought that Rayleigh's method had great advantages. In reply to Dr. A. Russell, he also stated that the phase of the harmonic components could be determined, but with an ambiguity of half a period. This ambiguity could be removed by a rough integration over a portion of the curve.

Mr. J. E. SEARS asked if all the periods given by this method of analysis were multiples of a fundamental as in the cases of the ordinary Fourier's analysis.

The AUTHOR stated that the present method gave the periodogram of a function as it was given by Fourier's double integral theorem. In reply to a further question from the same speaker, he stated that, in analysing a column of numerical values, the method would give all the detail that was actually involved in the figures themselves.

Mr. A. EAGLE (communicated remarks) thought it ought to be clearly pointed out that if it was desired to analyse a function extending over a limited range into the ordinary Fourier's Series, it would be necessary to repeat this function end to end a very large number of times in the grating. If it was only inserted once in the grating we should get the Fourier Integral analysis of a function, equal to the given function within the given limits, but zero everywhere outside these limits, which was quite a different thing. For any periodic function stretching from $-\infty$ to $+\infty$, it was easily seen that Fourier's Integral broke down into a Fourier's Series. Obviously, in this case, the periodogram must vanish for all wave-lengths that are not aliquot parts of the wave-length of the function analysed.

The AUTHOR entirely agrees with Mr. Eagle's remarks. So far he has not contemplated the application of his method to strictly periodic functions, because the coefficients of their various harmonic constituents can be so readily obtained by mechanical integration or otherwise.

XXVIII. *Some Experiments to detect β-Rays from Radium A.*

By W. Makower, *M.A.*, *D.Sc.*, *and* S. Russ, *D.Sc.*

Received March 26, 1913. Read May 16, 1913.

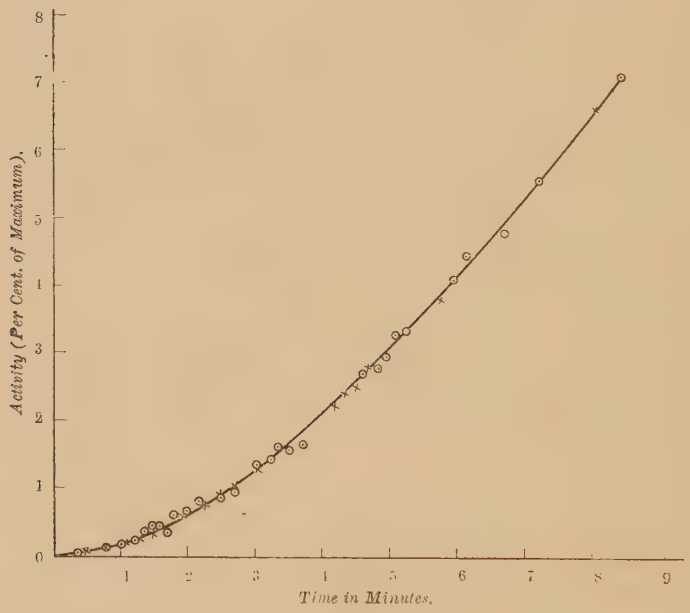
When an atom of radium A disintegrates, an α-particle is expelled which carries with it two positive atomic charges. At the same time, the radium B atom formed recoils with a single positive atomic charge. To account for these facts it is necessary to suppose that three negative electrons are expelled during the process. If these are emitted with a high velocity they should appear as β-rays capable of detection by the ionisation they produce or by their photographic action. On the other hand, they might consist of a slowly moving δ-radiation, which would escape detection by either of the above methods. The experiments described below, made in the hope of detecting β-rays from radium A, failed to reveal any such radiation. Experiments were first made to determine the rise of the β-radiation from a vessel filled with radium emanation carefully freed from the active deposit. If radium A emits β-rays, the radiation should increase much more rapidly from nothing than if no such radiation exists. A comparison of the initial stages of the rise curve with that to be expected if only radium B and radium C emit β-rays should, therefore, decide whether any β-rays are also emitted from radium A.

The experiment was carried out by exposing some radium emanation to a strong electric field for some hours to remove the active deposit formed, and then admitting the emanation through a plug of glass wool and through a tube immersed in liquid air into an evacuated space, one side of which was closed by a thin mica window. The thickness of the mica was equivalent to 4·3 cm. of air in its power of stopping α-rays. The mica window was placed opposite an aluminium window situated in the side of a β-ray electroscope; the two windows, together with the air between them, just sufficed to completely absorb the α-rays from radium C.

It has been shown by Fajans and Makower* that with a screen just thick enough to absorb the α-rays from radium C, the ratio of the ionisation produced by the β-rays from radium B to that produced by the β-rays from radium C is 1·33.

* Fajans and Makower, "Phil. Mag.," February, 1912.

Assuming this value, the variation with time of the ionisation produced by radium emanation freed from the active deposit can be calculated with the help of the table given by Moseley and Makower.* The curve calculated in this way is plotted in the figure. The experimental values obtained in two series of experiments are also shown, and will be seen to lie closely on the theoretical curve. The points in the two experiments have been made to fall on the theoretical curve at 8 minutes 2 seconds and 8 minutes 22 seconds respectively. This was done since it was found difficult to obtain with



Rise of Activity from Pure Emanation (β-rays).

sufficient accuracy the whole curve from the start of the experiment until the maximum activity was attained. The agreement between experiment and theory leaves no doubt that under the experimental conditions no β-rays entered the electroscope. This is in agreement with some recent observations by Danysz.† There still, however, remains the possibility of the existence of β-rays too soft to penetrate the absorbing material between the emanation and the electroscope.

* Moseley and Makower, "Phil. Mag.," February, 1912.
† Danysz, "Le Radium," January, 1913.

To decide whether any very soft β-radiation exists, experiments were made by one of us (W. M.), using a photographic method. A bare wire was exposed for five minutes to radium emanation so as to collect on its surface a deposit of radium A. The wire was then quickly mounted in vacuo in a magnetic field in front of a narrow slit, and the magnetic spectrum of the radiation passing through the slit was investigated. Although photographic records of β-radiation were obtained, these were traced to radiation from the small quantity of radium B formed from the radium A during the experiment; for the lines obtained coincided with those obtained with a wire exposed to the emanation for several hours and not mounted in the photographic apparatus until all the radium A had decayed. The lines detected were those observed by v. Bayer, Hahn and Meitner* and by Danysz,† having velocities 0·36, 0·41 and 0·63, expressed as fractions of the velocity of light. The method was thus shown to be sufficiently sensitive to detect any β-rays from radium A. The experiments indicate that any β-radiation from radium A, if it exists at all, must have a velocity less than one-tenth of the velocity of light.

ABSTRACT.

When an atom of radium A disintegrates an α-particle is expelled which carries with it two positive atomic charges. At the same time the radium B atom formed, recoils with a single positive charge. To account for these facts it is necessary to suppose that three negative electrons are expelled during the process. If these are emitted with a high velocity they should appear as β-rays capable of detection by the ionisation they produce or by their photographic action. On the other hand, they might consist of a slowly moving δ radiation which would escape detection by either of the above methods. The experiments, which were made by both methods in the hope of detecting β-rays from radium A, failed to reveal any such radiation.

DISCUSSION.

Prof. C. H. LEES asked if Dr. Russ thought that the explanation was that the radiations were so soft that they could not be detected.

Dr. RUSS thought that this was unlikely, as they would probably have been detected if present.

* v. Baeyer, Hahn and Meitner, "Phys. Zeitschr.," 12, p. 1099, 1911.

† Danysz, *loc. cit.*

XXIX.—*Dust Figures.* *By* J. Robinson, *M.Sc., Ph.D.*

Received May 1, 1913. Read May 16, 1913.

The conditions for ripple formation between the nodes of a Kundt's tube were investigated by König over 20 years ago,* and he put forward a satisfactory theory to account for them. He first of all explained why the ripples are generally obtained only with such heavy powders as sand and emery powder and not with lycopodium. He showed that particles of very fine powder are carried along with a stream of air and hence soon collect at the nodes. Heavy powders are not appreciably carried along by streams of air, and in the Kundt's tube they can be treated practically as at rest in the streams. He then put forward the theory that the ripple formation is due to the known hydrodynamical forces between particles in a stream. Considering a pair of particles and supposing them spheres, there is a repulsion between them if the line joining their centres is parallel to the stream and an attraction if perpendicular to the stream. By analogy with the case of iron filings in a magnetic field he accounted for the formation of ripples.

It has been suggested† that these forces are not in themselves sufficient, and that some constraint must be introduced, such as, for instance one due to viscosity. In the present Paper it will be shown that no such constraint is necessary.

König's formula for the component of the force between two spheres of radius R parallel to the direction of flow is

$$F=-\frac{\pi}{2}\sigma R^6\omega^2\frac{\cos\theta\,(3-5\cos^2\theta)}{r_0^4}, \quad . \quad . \quad . \quad . \quad (1)$$

where σ is the density of the air,

r_0 is the distance apart of the centres,

ω is the velocity of the fluid,

θ is the angle between the line joining the centres and the direction of the flow.

As this force depends on ω^2 it was possible to explain the fact that the distance apart of two ripples varies with the distance from a node; ω^2 is greatest at the antinode, hence the force between any two particles is greatest there, other things being

* "Wied. Ann.," XLII., 1891, pp. 353, 549.

† Cook, "Phil. Mag.," 1902, May, p. 471.

the same. By a direct application of this formula* it was shown that if a_{12} is the distance apart of two ripples at the antinode and $a_{r.r+1}$ the distance apart of two ripples at a distance k from the antinode, then

$$\frac{a_{r.r+1}}{a_{1.2}} = \left(\frac{\omega_r}{\omega_1}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where ω_1 and ω_r are the velocities at these places. In the case of the Kundt's tube this formula becomes

$$\frac{a_{r.r+1}}{a_{1.2}} = \cos^{\frac{3}{2}} \frac{\pi}{2} \frac{k}{l}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where $2l$ is the distance between two nodes. This formula (3) was found to agree with actual measurements made on the ripples.

Another point which was shown to be in harmony with König's theory was that the distance apart of the ripples is a function of the intensity of the sound. The intensity was varied by stroking the rod as violently and as lightly as possible. It was found that the mean distance apart of two ripples is greater the greater the intensity of the sound.

The same characteristics have been shown to hold for the dust figures produced by exciting an electric spark either at the end of an open tube or at a hole at the centre of a horizontal plate. In the case of lycopodium powder on a plate it was found that the ripples get nearer together as their distance from the spark increases. This suggested at once that the figures are produced by the sound of the spark. The velocity of the air diminishes as we recede from the spark, and by analogy with the Kundt's tube figures this at once accounts for the ripples getting nearer together. The formula (2) was shown to agree well with actual measurements on some figures. The same formula was obtained independently by Marsh and Nottage† and also verified. It is not easy to find definitely how the intensity of sound diminishes along the surface of a plate. König‡ arranged a spark so that the intensity of the sound would fall off according to the inverse distance law. This was arranged by sending the disturbance between two parallel plates at a distance of 3 mm. or 4 mm. apart. König

* Robinson, "Phil. Mag.," April, 1910, p. 476.

† "Phys. Zeit.," 1911, p. 439. "Proc." Phy. Soc. of London, XXVII., Pt. IV., 1911.

‡ "Phys. Zeit.," XII., 1911, p. 993.

found that formulæ (2) gave quite a satisfactory explanation of the dependence of ripple pitch on distance from the spark.

Ripples produced in an open tube by a spark at one end also show characteristics which have been explained by König's theory. The ripple pitch depends on the intensity of the sound of the spark in a way required by that theory, and is not intimately bound up with the wave-length of the short waves emitted by the spark.

König's formula (1) for the force between two small spheres was deduced on the assumption that their distance apart is large compared with the radius of either of them. In the case of dust figures the small spheres are so close together that doubts may be entertained as to whether the formula really applies to this case. In deducing equations (2) and (3) the assumption was made that König's formula (1) holds for the particles very close together. The total force of one ripple on a particle at the centre of a neighbouring ripple was obtained by summing the forces of each separate particle on it. The close argeement of these formulæ with experiment shows that the assumption is not far from the truth. König* has recently devoted some attention to this point, and has come to the conclusion from some experiments he made that his formula (1) expresses very closely the force between two spheres, quite independently of whether they are far apart or near together. This fact enables us to place more confidence in the formulæ (2) and (3).

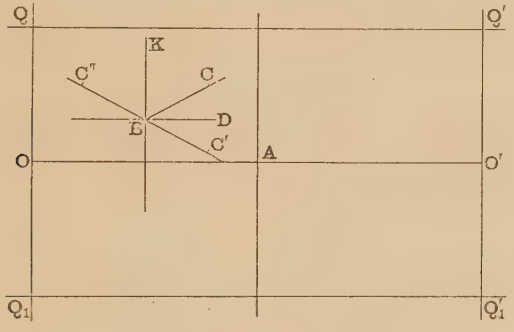
There is another point of resemblance between the Kundt's tube and the electric dust figures, and that is that when under the influence of the sound they stand up in sharp walls. This at once distinguishes them from the sand ripples on the sea shore. This wall formation can be readily seen with the Kundt's tube when a heavy powder is used. It is also noticeable with lycopodium if the rod is stroked lightly. With the electric dust figures one of two methods can be used to demonstrate the wall formation. They can either be examined by a microscope or projected on a screen. These walls are so sharp that it is desirable in making measurements on ripples to take instantaneous photographs of them whilst under the influence of the sound.

König's theory is quite capable of offering an explanation of this wall formation. We will first of all, however, devote

* "Phys. Zeit.," 1911, p. 991.

our attention to find whether the hydrodynamical forces are sufficient to account for the formation of ripples from a uniform distribution of powder.

First consider the Kundt's tube. Let us suppose that the bottom of the Kundt's tube is plane and that sand is strewn uniformly over it. As soon as sound waves pass along the tube each particle of powder will be acted on by forces from all the other particles in the distribution. We will suppose that the tube is adjusted for resonance and restrict ourselves to the space between two nodes. Let QQ′ and Q'_1Q_1 represent the sides of the tube, their distance apart being $2b$ $(=2 \cdot OQ)$. Let OO′ represent the axis of the tube, the nodes



being at O and O′ and the antinode at A. Between O and A the velocity of the air varies according to the equation

$$\omega = \omega_0 \cos \frac{\pi}{2} \frac{x}{l},$$

where ω_0 s the velocity at the antinode A,

$2l$ is the distance apart of two nodes,

x is the distance measured from A.

König's formula (1) applies only to the case of ω constant and must be modified for application here. The necessary modification* is

$$F = -\frac{\pi}{2}\sigma R^6 \omega\, \omega' \frac{\cos\theta(3-5\cos^2\theta)}{r_0^4}, \quad . \quad . \quad . \quad (4)$$

where ω and ω' are the velocities at the positions occupied by the two particles.

A particle at B is acted on by all the particles in the distribution immediately sound waves are started. The resultant

* "Phil. Mag.," April, 1910, p. 476.

force on B would be zero if the distribution is uniform, the velocity the same throughout, and if B is far from the sides of the tube. For if we consider either the component of the force parallel to the axis or perpendicular to the axis, we can always find two particles symmetrically situated with regard to B whose attractions or repulsions on it balance each other. Considering the components of the forces which are attractive we must choose two particles C and C′ situated on opposite sides of the line BD which is parallel to the axis of the tube. For the repulsions we must choose the two particles C and C″ situated symmetrically with regard to a line BK perpendicular to the axis of the tube. We are thus unable to account for the ripple formation without removing the restriction that the velocity is uniform.

If we also remove the restriction that the point B is far from the sides of the tube we find that the attractions on it are not symmetrical unless it lies on the axis of the tube. There is thus an initial tendency for a particle at B to move towards the axis of the tube.

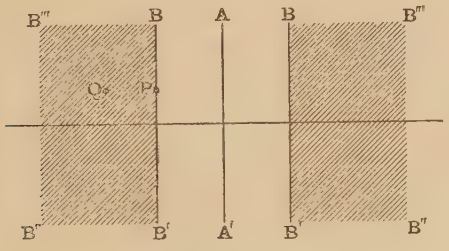
We will also find that there is a resultant force on this particle parallel to the axis of the tube if it is not at the antinode. For if B is some distance from the antinode, two particles symmetrically placed with regard to the line BK have not equal and opposite effects on B. As the velocity at C on the antinode side of B is greater than at C″ the repulsion of C on B is greater than that of C″ on B by equation (4). The same remark applies to all pairs of particles symmetrically placed with regard to BK. Hence there is a resultant force on a particle at B tending to move it to a node. Immediately sound waves are set up in the tube the uniform distribution of powder between two nodes is upset the particles which were initially at an antinode remain there, and all other particles are displaced in the direction towards a node. Thus we account for a ripple forming at the antinode.

This tendency of particles to a node was demonstrated by preparing a board of width just less than the diameter of the Kundt's tube. A ridge was made on it by making a fine saw cut across it and fastening in it a narrow strip of copper foil. The ridge projected $\frac{1}{2}$ mm., 1 mm. and 2 mm. above the surface of the board on different occasions. Powder was strewn uniformly over the board, which was inserted in the tube. When the rod was stroked the powder jumped violently over the ridge, but always in the direction from an antinode towards

a node. In this way sand grains could be made to jump nearly a centimetre above the surface of the board.

The ridges performed another function and showed that they take the place of ripples. A ridge acted just like a ripple in so far as there were ripples on both sides of it, and when under the action of the sound no powder was between the ridge and the neighbouring ripples. This function of a ripple of clearing the space between itself and a consecutive ripple of powder is characteristic, and can easily be observed whilst the rod is being stroked.

We have accounted for a ripple forming initially at the antinode. This ripple will repel particles near it, thus causing a clear space on each side of it. The distribution of powder is now somewhat as follows : At an antinode there is one ripple, AA′, on both sides of it a clear space, and then the particles are heaped together in some way at B, B′, B″, B‴. The width of the clear spaces AB will depend on the intensity of



the sound, for the forces exerted on particles by the antinode ripple depend on the intensity. Now this width AB fixes the distance apart of the ripples.

We can now readily account for the formation of the other ripples. A particle at P will be subjected to forces from the antinode ripple, and to those in the opposite direction from particles on the node side of it. These forces will be equal for a certain distance, AB, and then all particles at this distance will form into a second ripple. This in its turn will make a clear space on both sides, and so on. In this way we explain the initial formation of ripples purely on the fact of the hydrodynamical forces between two spheres in a moving fluid. It is unnecessary to postulate any other constraint on the particles, such as, for instance, friction.

Hence it is possible to obtain figures in the Kundt's tube without the assumption of any other forces whatsoever. In

the case of the electric dust figures the velocity of the air along the tube or across the plate does not vary from zero to a maximum, *i.e.*, there are no nodes, and we must consider the whole distribution of powder. We can account for the formation of ripples in these cases by removing the restriction of a uniform distribution of powder, which is of course a very improbable distribution. A small discontinuity at any point causes the hydrodynamical forces to lose symmetry. By similar reasoning to the preceding for Kundt's tube this causes first one ripple and then others to form.

Once the ripples are formed we can consider how the particles in each ripple distribute themselves. Between any two particles in the tube there may be either an attraction or a repulsion. For two particles in the same ripple there is an attraction varying inversely as the fourth power of the distance between them. The particles of one ripple hence take up positions so that the mutual forces between them cannot make them move any more; in other words, the particles adjust themselves so that the line joining the centres of any two particles is perpendicular to the stream lines. Hence we see how the ripples are straight walls when under the action of the sound waves.

König's theory is thus found capable of explaining all known facts about the Kundt's tube and electric dust figures. The theory accounts as well for the necessary constraints for ripple formation as for the forces between the particles. Hence it is not necessary to look for constraints in the action of viscous forces, although, of course, the most rigid discussion of the figures must take these into account.

East London College, April 15, 1913.

ABSTRACT.

The ripple formation in Kundt's tube was first satisfactorily explained by W. König in 1891. His theory was based on the hydrodynamical forces between two particles in a stream. All the known facts about these figures fall into line with the theory. The distance apart of the ripples increases with the intensity of the sound, and also from the node to the antinode there is a variation of the ripple pitch as required by the theory.

Certain measurements on dust figures produced by an electric spark have shown that these figures also can be explained in a similar way to the Kundt's tube figures.

It was suggested by Cook some years ago that viscosity must be introduced in order to account for the formation of ripples. The

author shows that it is possible to account for ripple formation without introducing viscous forces at all. These will undoubtedly play some part, but more as a disturbance than as a help to the formation. In the case of the Kundt's tube figures there is a variation of velocity of the air from a node to an antinode, which produces a variation in the forces, and this causes the powder to lose its uniformity of distribution and to form ripples. The necessary constraints for the ripples are forthcoming without the introduction of viscosity.

DISCUSSION.

Prof. C. H. Lees thought that Dr. Robinson's Paper filled up a gap in the subject. He had been astonished to see the magnitude of the forces that seemed to act on the grains in some of the experiments Dr. Robinson had shown him.

Prof. J. W. Nicholson thought Dr. Robinson's explanation was extremely convincing. He would like to know what kind of accuracy had been obtained in verifying the formula for the variation of pitch of the fringes from node to antinode, as it was deduced on the assumption that the force between any two particles was uninfluenced by the remaining ones. This correction would probably not be large in practice. The assumption that viscosity was the important factor in the phenomena was obviously untenable.

Dr. A. Russell remarked that it seemed from the present Paper that the production of dust striations with an electric spark could not be regarded as proof that the discharge was oscillatory, as it often was.

Mr. F. E. Smith asked if Dr. Robinson had ever tried to get the ripples with a light powder such as lycopodium powder. Although the force diminished very rapidly with the radius of the particles, since, in practice, the distance apart would also diminish, the actual force per unit mass acting on the particles might be about the same.

The Author, in reply to Prof. Nicholson, stated that the accuracy with which the formula was verified so surprised König that he set some of his students to measure the force between two spheres. This formula held for distances apart that were comparable to the radius of the spheres. He did not think that the production of striations with an electric spark proved anything as to the oscillatory character of the discharge. In reply to Mr. F. E. Smith, he stated that it was quite possible to get the ripples with lycopodium powder if the intensity of the sound was kept very low. König investigated this point and showed that light powders were carried along and collected at the nodes. When this was prevented the ripples were obtained.

XXX. *Vibration Galvanometer Design.* *By* H. F. Haworth, *Ph.D., M.Sc., B.Eng., A.M.I.E.E.*

Received May 2, 1913. Read May 16, 1913.

In a previous Paper by the author ("Proc." Phys. Soc., Vol. XXIV., Part IV., June 15, 1912) it was shown that in a moving-coil vibration galvanometer the maximum sensibility as a voltage detector is obtained when the flux through the coil is so adjusted that at any time the back E.M.F. of the coil is equal to, and in phase with, its ohmic resistance drop (CR).

The losses in a vibration galvanometer may be divided into two parts, the electrical losses $=C^2R$, and mechanical losses, such as air and molecular friction.

The maximum deflection for a given applied voltage is produced when the electrical losses are equal to the mechanical losses, the efficiency of the instrument then being 50 per cent. In this case we may write mechanical losses $=C^2R$, so, for a constant applied voltage and resistance, the maximum amount of power capable of being used in producing motion is constant.

As the frequency increases the losses for a given deflection increase, and so the deflection for a given voltage will decrease, but in order to obtain the maximum mechanical output the back E.M.F. must be maintained constant.

If the losses with a given system vary directly as the frequency and deflection, then the deflection for a constant input will vary inversely as the frequency, but the back E.M.F. produced is proportional to the product of the frequency and deflection, *i.e.*, it will remain constant, which is the condition required. If the losses increase at a greater rate than the first power of the frequency, then, on account of the diminished deflection, the flux in the gap must be increased.

The vibration galvanometer might be used as a delicate testing machine in order to observe the behaviour of different metals under alternating forces of different values and frequencies. The mechanical input under the maximum sensibility conditions is exactly known, and air friction could be allowed for or eliminated by working in vacuum, and so the molecular losses under the given conditions could be determined.

If the mirror of the instrument had no mass the frequency would be inversely proportional to the length of wire. Now the back E.M.F. is proportional to $L \times B \times f$, where L is length of wire, B is flux density and f is frequency, *i.e.*, the back E.M.F. is proportional to B, $L \times f$ being constant, so alteration of length under the above circumstances would not necessitate an alteration of flux, but, owing to the mass of the mirror, the frequency increases at a slower rate than the inverse length, and, as the wire is shortened, the flux density must be increased to keep the back E.M.F. at its proper value.

If the instrument is used at its highest frequency the effective length of the wire is short compared with its total length (it may be only one-seventh), the losses are high and its sensibility is low. The mass of the mirror is now very important and it causes the working length of the wire to be much shorter than would be required if the mirror had no mass. The result is that at high frequencies the gap flux density must be high compared with that used for low frequencies.

The resistance of the wire of an ordinary Duddell vibration galvanometer is of the order of 140 ohms, and of this only about 20 ohms is effective at high frequencies, so the remaining 120 ohms may be considered as an added resistance. Not only must the flux density be increased on account of the previous reasons, but it must also be much larger than it need be on account of the resistance of the non-working part of the wire, in fact, the total increase of flux necessary for maximum sensibility at high frequencies is such that it is not easily obtainable with the present design.

Two improvements immediately suggest themselves :—

1. Cut out the non-working part of the wire, substituting for it wires of low resistance.
2. Shorten the length of the pole pieces.

The flux density required for long strings (9 cm. or 10 cm.) is about from 4,000 to 6,000 lines per square centimetre for phosphor bronze, and this may just as well be obtained by running a gap, say, 3 cm. deep at from 12,000 to 18,000 lines per square centimetre, as by having one of 9 cm. deep at 4,000 to 6,000 lines per square centimetre. The advantage comes in when the wire is shortened. If the total flux is kept constant the average working flux density of the wire increases as the length decreases until the ends of the wires are between the poles when the flux density remains constant.

The combination of the reduced resistance of the instrument with the increased flux density at the higher frequencies enables one, with the aid of a variable magnetic field, to obtain the maximum sensibility of the instrument over all its ranges with a more moderate size of electromagnet; in fact, for a wide frequency range, it would not be necessary to bring the bridge pieces between the pole pieces, thus allowing the length of the air-gap to be reduced and so making the necessary electromagnet still smaller.

The above alterations were carried out on the instrument previously used by the author; the original pole pieces were cut down from 10 cm. with square ends to 7 cm. with ends tapering to 4 cm. at the gap. Although the shape of the pole pieces did not lend itself to efficient working with a strong electromagnet, the old electromagnet, 7 cm. deep, used in the previous experiments being utilised, the increased flux density obtained was sufficient, in conjunction with the resistance alteration, to demonstrate the value of the improvements.

The superfluous wire was cut out at the top bridge by running the wire, after leaving the ivory bridge piece, under a small platinum-silver bridge fixed into the back part of the ivory bridge. The current was taken to the wires at the bottom bridge through two platinum-silver hooks rubbing on the wires. These hooks were soldered on to pieces of copper wire fixed into the ivory bridge; these pieces of copper wire projected on either side of the bridge piece each terminating in an eye which engaged on a thin stranded copper wire running parallel with the operating screw. These copper wires were carried at the top on small ivory studs fixed on the tube support at the middle of the galvanometer frame, and the bottom ends were soldered to lugs which also carried the working wires so that the non-working part of the wire and the copper flexible up to the bridge were always in parallel.

With the phosphor-bronze wire previously used the contacts made by the platinum-silver bridge pieces were not always good; the wire seemed to be patchy, so, as no new wire was at hand at the moment, a platinum-silver wire of the same diameter, 1·4 mils, was used. This was immediately successful, the resistance variation with length being very steady, the equation being $R = 8 + 5{\cdot}7L$, where L equals length of string in centimetres.

The resistance of the portions of wire from the ivory bridge pieces up to the platinum-silver bridge pieces was about 8 ohms;

this is about one-third of the resistance of the galvanometer with the wires in the shortest position used (2·5 cm.=22 ohms), and this could be still further reduced if the bridge pieces were re-designed for this kind of work.

A series of experiments was carried out in which different string lengths and different tensions were used. In each case the flux was varied from a maximum to 10 per cent. of the maximum, and, for each value of the flux, the millimetres per micro-ampere and the millimetres per milli-volt at a metre were determined. From these results were obtained the apparent resistance of the instrument and the efficiency as determined by the millimetres per milli-micro-watt at a metre. The direct-current sensibility was also obtained and the resistance was measured.

The method of testing was similar to that described in the previous Paper, the only alteration being the substitution of a Paul unipivot dynamometer in series with R_1, which not only measured the current through R_3 but also the voltage across R_2. This gave a much quicker working arrangement. A portable Wheatstone Bridge was also introduced into the circuit in order to measure the galvanometer resistance. The results are summarised in the Table given ; the first part of the results refer to the experiments previously made, using a phosphor-bronze wire, with some additional figures worked out, the wire being of constant resistance and working between pole pieces 10 cm. long. The platinum-silver wire readings were taken with the travelling contact bridges and the 4 cm. tapered pole pieces.

Comparing the phosphor-bronze and the platinum-silver wires under as near similar conditions as possible, taking the 9·5 cm. wires the following points will be noted :—

The maximum direct-current sensibility is greater in the case of the long string and long pole pieces than with long string and short pole pieces on account of the greater flux through the wire due to the lower reluctance of the circuit. As the string shortens the direct-current sensibilities decrease, but, owing to the fact that the flux density is constant in the first place, and is steadily increasing in the second case, with decrease of length until the bridge pieces enter the field, the falling off in sensibility takes place at a much greater rate with the long poles than with the short poles, and with a 6 cm. string the sensibilities are of the same order ; after this the advantage lies with the short pole pieces.

Wire used, &c.	Length, cm.	Tension.	Frequency per sec.	Maximum voltmeter sensibility, mm.-milli-voltmetre.	Average flux density for this voltmetre sensibility.	Resistance of galvanometer.	Apparent resistance for max. sensibility.	Mm. per 10^{-9} watt at a metre for max. sensibility.	Max. mm. per micro-amp. at a metre for max. flux.	Mm. per 10^{-6} watt D.C. max. field.	D.C. sensibility, mm. per milli-amp. at a metre.	Current magnification due to resonance.
1·4 mil phosphor-bronze	9·5	1	237	95	4,630		275	26·0	46	159	148	310
		2	295	81	5,340			22·2	44	64	93·5	470
		3	320	74	5,940			20·3	38	38·3	72·5	525
Pole pieces 10 cm. long	6·5	1	310	73	8,270			20·0	26	41·7	75·7	343
		2	410	53	8,650			14·5	22	10·3	38·2	576
		3	526	41	9,280			11·3	14·5	3·7	22·6	640
				No maximum sensibility. Value at max, flux density.		137·5	varies up to					
	2·5	1	520	17	14,000		200	3·4	3	0·9	11·1	271
		2	740	14·4	14,000		200	2·6	2	0·23	5·6	358
		3	900	10	14,000		160	1·6	1·6	0·10	3·8	424
1·4 mil platinum-silver not annealed	10	1	237	84	2,740	64·8	139·6	11·7	26·0	198	113	230
		2	315	76				10·6	21·6	83	73·5	294
		3	360	69·6				9·7	20·0	44	53·3	375
	8	1	250	67·2	3,440	52·9	105·8	7·1	16·0	154	90·0	79,
		2	392	68				7·2	15·2	39	45·5	158
		3	440	60·8				6·4	12·3	29	39·2	163
Pole pieces 4 cm. long	6	1	360	50·5	5,930	42·0	84	4·25	9·0	93	62·5	144
		2	494	46				3·87	8·7	22·5	30·8	282
		3	600	48·5				4·08	8·2	9·5	20 0	410
	4	1	495	44·5	10,000	31·5	63	2·80	5·3	26	28·6	185
		2	675	40				2·52	5·2	8·0	16·0	325
		3	860	36·7				2·31	3·0	3·9	11·1	270
	3	1	595	29	11,500	26·0	52	1·51	2·3	15·4	20·0	115
		2	800	29					1·85	5·0	11·8	157
		3	940	29					1·84	2·9	8·7	212
1·4 mil annealed Platinum-silver	9·5	1	237	98	2,260	63	126	12·3	28·4	170	103	276
		2	325	97				12·2	32·0	59·6	61·5	520
	2·5	1	740	37·5	11,550	23	46	1·73	2·0	6·4	12·1	165
		2	920	38·5				1·77	2·6	3·0	8·4	310

The same remarks apply to the figures for the alternating-current sensibility, though to a considerably smaller degree on account of the fact that we are now dealing with a vibrating system, and the losses in the phosphor bronze are apparently smaller at low frequencies and long strings than with platinum silver. At high frequencies with short strings the wires seem equally efficient; this is seen on examining the deflections per watt.

The mechanical losses for maximum sensibility are equal to $E \times I = \frac{V^2}{4R}$ or $\frac{V^2}{4} = R \times$ mechanical losses, so for a constant applied voltage the losses vary inversely as the resistance. Working at a point of maximum reception of energy for producing mechanical work, we have naturally a constant watt sensibility for a given deflection, because the losses for a given deflection and frequency are constant. If we increase the galvanometer resistance we must correspondingly increase the flux density to still keep at this point; the back E.M.F. must increase correspondingly with the resistance. The sensibility of the instrument as an ammeter is independent of the resistance, but its sensitiveness as a voltage detector varies inversely as the resistance. If the frequency of the moving coil be now increased, the power required to vibrate the mirror through the same angle as before is increased, but, as the maximum amount of power available for vibrating the system, considering a constant resistance instrument, is still the same, the deflection, therefore, is decreased.

The back E.M.F. has still to be the same, the frequency is greater, so the flux may have to change to keep to the point of maximum sensibility. In the case of the phosphor-bronze wires the flux increased with the tension, but, using platinum-silver wire, the flux was practically constant with varying tensions. From the above and the consideration of the efficiency column, there is an indication that the losses in platinum silver increase at a slower rate with increase of tension than the losses in phosphor bronze.

The great advantage of a variable resistance instrument is seen at the higher frequencies. At maximum sensibility the watts input to the 2·5 cm. string is about three times that of the 10 cm. string; this is as it should be because as the frequency increases the sensibility diminishes. The short pole pieces now show to advantage because we are still able to put the total flux through the moving system and use it instead of

having it acting on the inoperative part of the wire. The efficiency of this particular system is now equal to the phosphor bronze, the current sensibilities are of the same order, but the voltmeter sensibility of the instrument is about three to four times greater with platinum silver than with phosphor bronze. With a frequency range of about 4 to 1 the voltmeter sensibilities have dropped in the case of the constant resistance instrument in the ratio of 10 to 1 and with the variable resistance instrument in the ratio of 2·5—3 to 1, which is a great improvement.

Owing to the lower initial resistance of the platinum-silver wire, its voltmeter sensibility at low frequencies, although its losses are greater, is equal to that of the phosphor-bronze wire, while at high frequencies it is much superior.

A few readings were taken with an annealed platinum-silver wire. The original wire was not quite straight and could not be straightened by increasing the string tension ; this caused the mirror to point in various directions as the bridge pieces were moved, so another wire was taken and heated to red heat by a current of 0·3 ampere, while a small weight kept the wire stretched. This cured the mirror trouble and gave an improved moving system. It is interesting to note that the efficiency of the short tight platinum-silver wire is equal to that of the short phosphor-bronze wire.

The flux densities are average values over a field 2 mm. wide, so are only comparative and do not necessarily represent the flux density through the wire.

I have to thank Mr. F. W. Andrews of the City and Guilds (Engineering) College for his skill and ingenuity in making the necessary alterations.

ABSTRACT.

1. The maximum amount of power available for vibrating the moving system of a vibration galvanometer of the moving-coil type is $V^2/4R$. As the frequency of the instrument is raised the losses increase rapidly, so it is an advantage to be able to increase the useful power input per unit voltage. To do this the resistance of the instrument must be decreased. This can be done in a galvanometer of the Duddell type by leading the current in and out at the bottom bridge and short-circuiting the wires at the top bridge, and it results in a great increase of sensibility. A lower resistance also requires a lower flux density.

2. Owing to the losses in the moving system increasing at a greater rate than the first power of the frequency, and that the frequency of the system increases at a slower rate than the reciprocal length of the

string on account of the mass of the mirror, the flux density must be increased as the frequency increases in order that the back E.M.F. of the moving system may always be half that of the applied P.D. As the losses are low at low frequency and the mass of the mirror is not large, then, compared with the mass of the wire, the flux density required is moderate; but at high frequencies the flux density required is large. In order to obtain this result economically it is convenient to make the depth of the poles small compared with the maximum length of the wires. This gives a sufficient field for the long wires, and for short wires one is able to obtain the necessary flux density because the total flux can still be put through the moving system.

3. A combination of 1 and 2 makes a very satisfactory instrument with a much flatter voltmeter-sensibility-frequency curve than obtained, usually.

DISCUSSION.

Mr. W. Duddell remarked that the Paper dealt chiefly with varying the length of the vibrating wires, but if it was required to make a really efficient galvanometer for high frequencies it was best to re-design the whole instrument, as the distance apart of the wires, the scale distance, size of mirror, &c., all ought to be altered. These various factors could be calculated beforehand, and the instrument relied upon for behaving according to the calculations. When a very low resistance was wanted silver wires could be used. He was rather afraid that sliding contacts would not be found satisfactory. It was quite true that a strong field in the centre of the wire was as good as a weak one for the whole length, but a strong field necessitated using an electromagnet, against which he had a prejudice, for vibration galvanometers. This necessitated a battery to excite it, and not only made the insulation more difficult, but also greatly increased the capacity to earth which for high frequencies it was very necessary to keep low. In his own laboratory, if he connected one termin[illegible] a vibration galvanometer, tuned to the alternating electric light mains, to earth, he could get a very considerable deflection simply due to capacity currents. He was glad to see that Dr. Haworth had pointed out that the back E.M.F. of a vibration galvanometer behaved like a resistance since it was proportional to the applied E.M.F.

Mr. A. Campbell pointed out that it was not in all cases necessary that the instrument should have as great a voltage sensitivity as possible. In some cases it was current sensitivity that was required. This depended upon the bridge the galvanometer was used with. With respect to Dr. Haworth's suggestion that the vibration galvanometer could be used as a testing machine for studying the elastic behaviour of wires, he had thought the same himself for some time, but engineers replied that it was not the properties of wires that they wanted. These had become so altered during the process of drawing as to be useless to apply to other specimens. The tests would be useful as giving the properties of wires at high frequency. In this connection he drew attention to Kapp's machine for testing materials under rapidly alternating stresses. He asked if Dr. Haworth could give any information as to how the dynamical hysteresis varied with the angle of twist. He fancied it varied as a fairly high power. He believed that nearly all the hysteresis damping was due to the bending and not to the twisting of the wires. He had had the same experience as Mr. Duddell with respect to capacity currents.

Dr. A. Russell thought that a few more formulæ would have been desirable in the Paper. He remarked that the expression $V^2/4R$ for the energy absorbed was capable of very easy proof as follows: Taking i, e and

e_b as being the instantaneous values of the current, the applied voltage and the back E.M.F. respectively, we have $ie_b = \frac{e_b(e-e_b)}{R}$. Taking mean values this gave $W = \frac{VV_b}{R}\cos\alpha - \frac{V_b^2}{R}$, where W was the power absorbed and V and V_b were the root mean square applied and back E.M.F.'s respectively, and α was the phase angle between them. This expression was very easily seen to give the maximum value of $\frac{V^2\cos^2\alpha}{4R}$, when $V_b = V\cos\alpha$. When adjusted for resonance $\cos\alpha$ became unity.

The AUTHOR, in reply to Mr. Campbell, stated that he had not worked out the hysteresis law.

XXXI.—*Electrothermal Phenomena at the Contact of Two Conductors, with a Theory of a Class of Radiotelegraph Detectors.* *By* W. H. Eccles, *D.Sc.*

Received May 16, 1913. Read May 30, 1913.

When an electric current is caused to pass across the interface between a pair of conducting masses, heat is in general liberated or absorbed in accordance with the law of Peltier. When the masses are in contact over a very small area, as, for example, when a cylinder of graphite is laid across a copper wire, there may be, in addition, appreciable generation of heat in accordance with the law of Joule. If the substances constituting the contact are bad conductors of electricity and of heat, and if they stand far apart in the thermoelectric series, the phenomena arising when a current is forced across the joint become very striking, for in such circumstances relatively large amounts of heat may be developed, the heat is conserved, and therefore the thermoelectric effects enhanced.

It is evident that the thermoelectric forces called up by the local heating may assist or may oppose the E.M.F. applied to produce the current and that the phenomena of asymmetric conduction at once arises. But besides the Joule and Peltier effects, the Thomson effect may contribute to the phenomena. In the case of bad conductors of heat the temperature gradients very near the contact will be very steep, and thus the Thomson effect will be localised in the immediate neighbourhood of the contact.

Further than this it is obvious that, on account of the temperature changes, the portions of the substances near the contact will suffer a change in the magnitude of their electrical resistivity. It has been shown* that this effect alone leads to remarkable and important results, and is sufficient to account for all the principal features of the single-point coherer used in wireless telegraphy.

The thermoelectric forces and the changes of electrical resistance that arise from differences of temperature are much greater in combinations of such substances as iron pyrites than in combinations of ordinary metals. A pyrites-lead couple yields an E.M.F. some 200 per cent. greater than a bismuth-

* "Proc." Physical Society of London, Vol. XXII., p. 869.

lead couple, between the same extremes of (ordinary) temperature; while the temperature coefficient of resistance of pyrites is about four times as great numerically as that of copper. But the thermal conductivity of pyrites is so very much smaller than that of lead that all these thermoelectric phenomena are greatly accentuated in the former case. Contacts between non-metallic conductors are of special interest for the reason that the bulk of the wireless telegraphy of the world is carried on by aid of detectors that consist of nothing else than a contact involving at least one non-metallic conductor.

The subject does not appear to have been discussed hitherto from the theoretical standpoint adopted in this Paper.

Thermoelectric Theory of a Contact.

The thermo-electric constants of such substances as pyrites, zincite, carborundum, &c., are not easy to measure accurately and their coefficients of increase of resistance with temperature are exceedingly difficult to determine. The author has made numerous determinations, and has found that all the materials examined follow with fair precision the ordinary thermo-electric law that their thermo-electric powers are linear functions of the temperature, and also that their temperature-resistance coefficients are all large and negative.

For a thermocouple formed of (say) pyrite and lead we may therefore assume the E.M.F. equation

$$e_\tau = \alpha\tau^2 + \beta\tau + \text{constant}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where τ represents the absolute temperature of the hot junction, α and β are constants, and e_τ is the E.M.F. when one junction is at temperature τ and the other at the absolute zero. The ordinary thermodynamic theory may legitimately be applied. Thus if σ represent the specific heat of electricity in the substance, P the Peltier effect and p the thermoelectric power at the temperature τ, the first and second laws of thermodynamics give, when applied to a cycle of infinitesimal range,

$$\sigma = \frac{d\mathrm{P}}{d\tau} - p,$$

and

$$\frac{d}{d\tau}\left(\frac{\mathrm{P}}{\tau}\right) = \frac{\sigma}{\tau}.$$

Whence we have

$$\sigma = \tau\frac{dp}{d\tau} = \tau\frac{d^2e}{d\tau^2}.$$

If we represent the absolute temperature of the surroundings and of the cold junction by z, and the temperature of the hot junction by $\theta+z$, we find from these equations that at the temperature of the surroundings

$$\sigma=2az,\ p=\beta+\sigma,\ \mathrm{P}=pz,$$

so that it is easy to calculate the thermoelectric quantities from the experimental equation (1).

Further, we obtain from (1)

$$\begin{aligned} e_\theta &= a\theta^2+(\beta+2az)\theta \\ &= p\theta+\theta^2/2z \\ &= \theta(\mathrm{P}+\tfrac{1}{2}\sigma\theta)/z. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2) \end{aligned}$$

When instead of lead a substance is used of which the thermoelectric properties are represented by σ', p', P', the E.M.F. in the circuit, when the junctions are at temperatures z, $\theta+z$, is

$$e_\theta=\theta\{\mathrm{P}-\mathrm{P}'+\tfrac{1}{2}(\sigma-\sigma')\theta\}/z. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Now, suppose the circuit made up of these two conductors

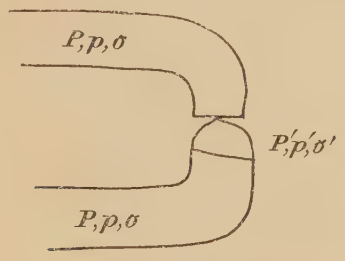


FIG. 1.

to have one junction forming a loose contact, the other a very good contact, as indicated in Fig. 1, where one conductor may be regarded as a piece of pyrites and the other as a piece of metal. Let an E.M.F. be applied to the circuit in any manner so as to produce a current in the direction opposite to the E.M.F. that would be produced by heating the contact. The heat liberated near the contact, say h, is, per absolute unit of electricity flowing across the contact, equal to the thermoelectric force e_θ plus the heat absorbed at the cold junction. That is

$$h=e_\theta+\mathrm{P}-\mathrm{P}'.$$

This heat tends to be concentrated in a small volume of the substance near the contact, but is dispersed continually by thermal conductivity and radiation. We will assume that the

rate of loss of heat by these agencies is, as a whole, proportional to the excess of temperature of the junction over its surroundings. The rise of temperature causes an alteration of the electrical resistance of the joint; let the true resistance at any temperature θ above the temperature of the surroundings be expressed by $\rho(1+\gamma\theta)$, where γ is a temperature coefficient dependent on both the substances at the contact.

By equating the rate of accumulation of heat to the rate of liberation of heat by the electrical processes less the rate of loss of heat to the surroundings by purely thermal processes, we have, in mechanical units,

$$k\frac{d\theta}{dt}+m\theta=hy+\rho(1+\gamma\theta)y^2, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where y is the current in the direction defined as above, k is a constant involving the densities and specific heats of both substances and m is a constant involving the internal and external thermal conductivities of the substances near the contact.

Let the part of the circuit which suffers inappreciable temperature change possess the constant resistance r and inductance L, and let the applied E.M.F. be e, at any time t, then

$$\mathrm{L}\frac{dy}{dt}+(r+\rho)y+e_\theta=e. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

We have to deal in this Paper with a state of affairs in which the rate of variation of current and applied E.M.F. is relatively slow. In this case we may use the equations of the steady state, namely,

$$m\theta=hy+\rho(1+\gamma\theta)y^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$(r+\rho)y=e-e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Substitute for e_θ from equation (3) above, eliminate θ, and we obtain an equation for the current y sent across the junction by the applied E.M.F. e. The equation is less cumbrous, however, if we obtain it in terms of the P.D. between the extreme parts of the conductors which vary in temperature. Regarding this as a portion of the applied E.M.F. and representing it by x, we have

$$x=e-ry+\mathrm{P}-\mathrm{P}'. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Then the current equation becomes

$$\frac{\sigma-\sigma'}{2m^2z}x^2y^2+\frac{\rho\gamma}{m}xy^2+\frac{\mathrm{P}-\mathrm{P}'}{mz}xy-x+\rho y+\mathrm{P}-\mathrm{P}'=0, \quad . \quad (9)$$

which may be written

$$ax^2y^2+cxy^2+bxy-x+\rho y+b'=0. \quad . \quad . \quad . \quad (10)$$

The constant coefficients a, c, b in this equation are, it will be seen, mainly dependent on the Thomson effect, the change of resistance with temperature, and the Peltier effect respectively. The term b' is always negligible except very near the origin of co-ordinates and will be put zero in all that follows. Subject to the proper interpretation of the variable x, the equation is that of the "steady current characteristic" of the combination of conductors. And since when b' is taken zero

$$x=e-ry, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

the steady current characteristic, as usually drawn from observations of applied E.M.F. and consequent current, is identically that obtained by applying to the curve drawn from the above equation a homogeneous shear of amount r parallel to the x axis.

Particular Cases.

Case A.—$a=b=0$.

The coefficient a is proportional to the difference between the Thomson effects in the two conductors at the absolute temperature z; if they are equal it vanishes. The coefficient b is proportional to the Peltier E.M.F. at the junction at temperature z; it vanishes if the neutral temperature of the substances happen to be z. For thermoelectric effects to be wholly absent a and b must both be zero. We are then left with the cubic

$$cxy^2-x+\rho y=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

which has been discussed in Papers* by the present writer. The quantity c depends mainly on γ, the coefficient of increase of resistance with rise of temperature. The most interesting cases arise when γ is negative, as it nearly always is in contacts including a native crystalline oxide or sulphide, or carborundum. In fact, this equation explains the action of the single-point coherer fairly perfectly; and thus, when in equation (10) this quantity c is finite and negative, we may say that this "coherer action" is accompanying the thermoelectric effects.

The kinds of curve that may be yielded by pure coherer action are shown in Figs. 2 and 3. In Fig. 2 the equation is

$$c'xy^2+x-\rho y=0,$$

* "On Coherers," "Proc. Phys. Soc.," Vol. XXII. "On an Oscillation Detector," "Proc. Phy. Soc.", Vol. XXII.

when c' is put for $-c$. The curve of Fig. 2 has only one real asymptote, the axis of y. In order to exhibit the curve con-

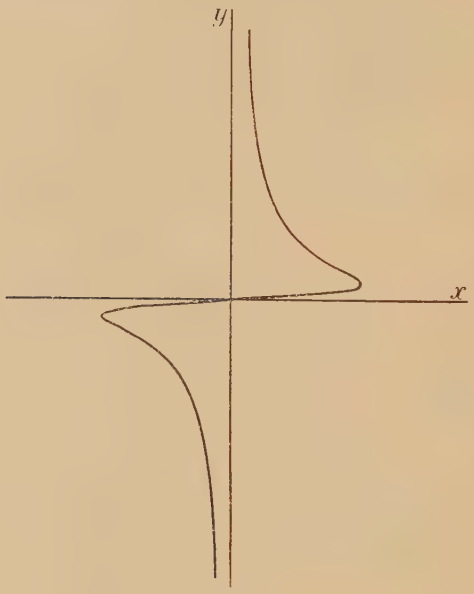


Fig. 2.

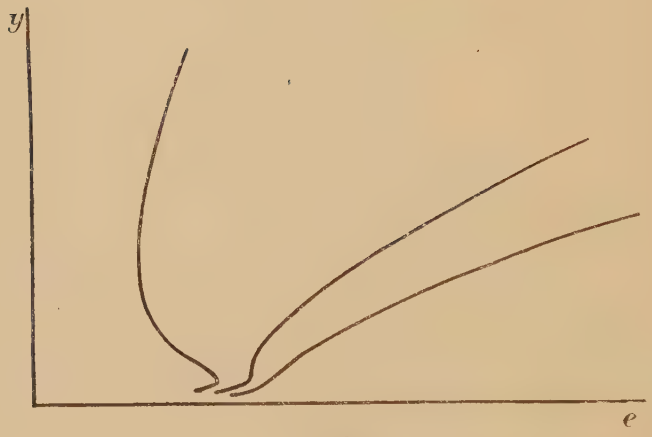


Fig 3.

necting applied E.M.F. and resulting current the curve must be submitted to shear of amount r, where r is the non-varying

part of the resistance of the whole of the circuit containing the contact. In this way we obtain curves of the kind shown in Fig. 3, which resemble those obtained* from a contact consisting of two pieces of galena. There are two vertical tangents if $8r<\rho$ (case I.) and none if $8r>\rho$ (case III.). The two vertical tangents coincide if $8r=\rho$ (case II.). Here ρ is the starting value of the variable part of the true resistance of the contact.

The curve I. shows the existence of instability; near the right-hand vertical tangent a very small increase of E.M.F. will produce discontinuously a great increase in the current; case III. is quite stable and illustrates a "self-restoring" coherer.

For contrast, the curve corresponding to the case when resistance increases with rise of temperature is shown in Fig. 4. The equation is

$$cxy^2-x+\rho y=0.$$

The part of immediate physical interest is that passing through the origin. It shows that, as the current increases, greater and greater increments of E.M.F. are needed to produce an assigned increase of current, as might be concluded without calculation. This is the kind of curve obtained in any circuit in which a conductor with positive resistance coefficient becomes heated; for example, a fuse wire raised towards fusion in air. The curve of case III., Fig. 3, arises when the heated conductor possesses a negative coefficient—*e.g.*, the carbon filament of a lamp.

The contact or aggregate of contacts that occurs between the carbon brushes and the commutator of a dynamo affords a striking instance of the phenomena. Many of the known properties of brush contacts have been explained by W. H. F. Murdoch † on these lines.

Case B.—$a=0$, $c=0$.

When the specific heats of electricity in the two conductors are equal $a=0$, and when there is no coherer action $c=0$. In this case the equation reduces to

$$bxy-x+\rho y=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

which gives the hyperbola shown in Fig. 5. Only the branch

* "On an Oscillation Detector," *loc. cit.*

† Murdoch, "Commutator Resistance and Energy Losses," "Electrica Review," June 16, 1911.

through the origin appears to be physically relevant. When sheared the equation is

$$bry^2-bey+e-(r+\rho)y=0.$$

This represents a hyperbola with its centre at the point $(r-\rho)/b$, $1/b$), and with asymptotes $by=1$, $b(e-ry)+\rho=0$; it is shown in Fig. 6. It is clear that the curve is not symmetrical about the origin, and, therefore, that the existence of the Peltier effect makes the contact a rectifier of alternating current of any amplitude.

Case C.—No coherer action. $\gamma=0$, and, therefore, $c=0$.

In this case the full effects of thermoelectric action, free from coherer action, are brought out. The equation is

$$ax^2y^2+bxy-x+\rho=0. \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Two sub-cases arise—namely, a positive and a negative.

1. *a Positive.*—This sub-case is shown in Fig. 7. The branch through the origin shows that the contact will act as an excellent rectifier, unless, indeed, the invariable part r of the resistance is relatively very large; for even after considerable shearing the curve will have its relevant negative portion much steeper than the positive portion. The contrast with the last case (Thomson effect zero) is especially noteworthy. In that case the difference of gradient on the two sides of the axis of y, after shearing, is not nearly so pronounced as in the present case.

2. *a Negative.*—When a is negative we obtain the curve shown in Fig. 8. It has its steepest part in the positive quadrant, and the points of inflexion near the origin indicate that the rectification is more pronounced than in the last case. Thus we may expect a combination in which a is of opposite sign to b to behave as a good detector for the purposes of wireless telegraphy, other things being not unsuitable. The meaning of this condition for good rectifying properties may easily be expressed by aid of the diagrams of thermoelectric power in Figs. 9 and 10.

In both figures the Peltier E.M.F.s P, P′ divided by the absolute temperature z are represented by zp and zp'; and the specific heats of electricity σ, σ' divided by the temperature z are represented by the gradients of the lines at the temperature z. Now

$$a=\frac{\sigma-\sigma'}{2m^2z} \quad \text{and} \quad b=\frac{P-P'}{mz}.$$

Hence in Fig. 9 both a and b are positive, while in Fig. 10 a is negative and b remains positive. In the one case the neutral temperature is below z, in the other above z. This is to be

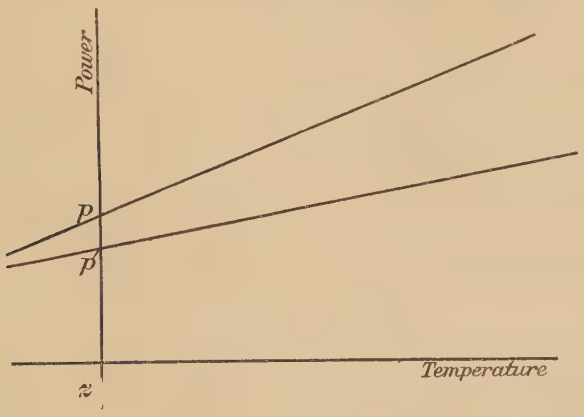


FIG. 9.

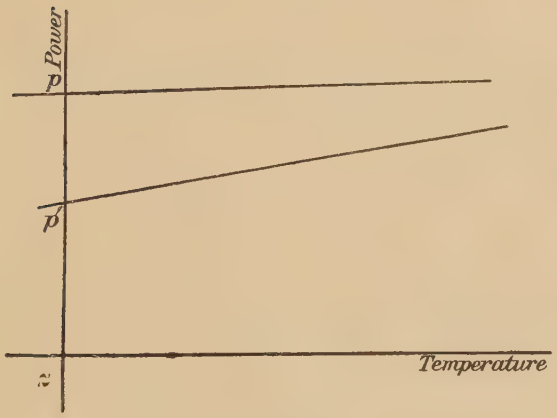


FIG. 10.

seen also from the familiar fact that the neutral temperature T is given by

$$T = -\frac{P - P'}{\sigma - \sigma'},$$

which in our notation is written

$$T = -\frac{b}{2ma}.$$

In words the theorem now arrived at is that the best rectifying

contact obtained by purely thermoelectric effects is that in which the neutral temperature (or "temperature of inversion") for the pair of substances forming the contact is above the temperature of the cold junction.

Case D.—$c=0$ and $b=0$.

The above conclusion suggests an inquiry as to the rectifying properties to be expected of a contact composed of two substances which have the same Peltier E.M.F. with respect to lead and still have no coherer action. The equation is now, c being still zero,

$$ax^2y^2-x+\rho y=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Two sub-cases arise, a positive and a negative.

1. *a Positive.*—The curve is given in Fig. 11. It indicates good rectification; a *negative* E.M.F. causes a greater flow of current than a positive E.M.F.

2. *a Negative.*—The curve is given in Fig. 12. In this case again there is good rectification, but now a *positive* E.M.F. causes the greater flow of current.

The principal difference between this case and the case C2 is that there is no point of inflexion near the origin in the present case. This again indicates that it is advantageous to have the neutral temperature high. In fact, referring back to case C, when b is not zero, and solving the corresponding equation as a quadratic in y, we have

$$y=\frac{-(bx+\rho)\pm\sqrt{(bx+\rho)^2+4ax^3}}{2ax^2}.$$

The vertical tangent to the curve is, therefore, given by the cubic equation for x

$$-4ax^3=(bx+\rho)^2.$$

First, taking a positive and of fixed value we see that the one real root of this equation is numerically greater the greater b, being $-(\rho/4a)^{\frac{1}{3}}$ in the limiting case $b=0$. Thus the greater the value of b the further does the vertical part of the physical branch of the curve depart from the axis of y. Since the neutral temperature is proportional to b, we may say that the rectification* produced by such a contact is greater the higher the neutral temperature. This is, moreover, the more marked the smaller the value of ρ. Now, taking a negative we see that the real positive root of the equation is greater the greater b,

* *See* Appendix p. 286.

and thus again the rectification is greater the higher the neutral temperature. Further, supposing that in turn b is fixed in value, the form of the equation shows at once that the smaller the numerical value of a the greater numerically is the relevant root in x. Thus again it appears that a high neutral temperature corresponds to good rectification. But in all these conclusions there is always to be kept in mind the practical condition that the root of the cubic shall be a value of x that is attainable experimentally.

Taken as a whole the argument establishes the theorem that the rectifying properties of a contact at which there is no coherer action are the more pronounced the higher (within practical limits) the temperature of inversion.

Case E.—The Resistance Coefficient not Zero.

The most interesting sub-cases occur when the Thomson effect in the circuit is zero. The steady current equation is

$$cxy^2+bxy-x+\rho y=0. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

There are three sub-cases :—

1. c Positive—that is, the resistance of the substances near the contact on the whole increases with rise of temperature. There are two real asymptotes parallel to the axis of x and on opposite sides of that axis. The curve (Fig. 13), is unsymmetrical about the origin and the contact possesses rectifying properties to some extent, but the asymmetry is smaller the greater c.

2. c Negative and $16c^2<b^4$. Here coherer action enters, but only to a relatively small extent. The two real asymptotes parallel to the axis of x are now on the negative side of that axis, and consequently the curve (Fig. 14) is very unsymmetrical about the origin. The contact will be a good rectifier, and will be a good detector of oscillations when submitted to a suitable positive E.M.F.

3. c Negative and $16c^2>b^4$. Here coherer action is predominant. The curve is shown in Fig. 15. The asymptotes parallel to the axis of x are imaginary, and the contact, when employed as a detector, is sensitive under a particular negative as well as a particular positive E.M.F. Its difference from a pure coherer would be seen experimentally by a difference in the numerical values of the particular positive and negative E.M.F.s that give maximum sensitiveness. The vertical tangents occur at values of x given by

$$\rho/(2\sqrt{-c}+b) \text{ and } -\rho/(2\sqrt{-c}-b).$$

GENERAL CASE.

We have now to analyse the general case when none of the constants in the steady current curve vanish. From what has gone before it may be expected that, broadly, the effect of the thermoelectric terms is to produce asymmetry and the effect of the resistance terms is to encourage symmetry in the physical portions of the curves.

The general equation is

$$ax^2y^2+cxy^2+bxy-x+\rho y=0. \quad . \quad . \quad . \quad . \quad (17)$$

By definition b is positive, and the positive direction of the current is that in which it must flow in order to absorb heat by the Peltier effect at the junction that remains at the temperature of the surroundings. The resistance ρ is essentially positive; the Thomson term a may be negative or positive, and c has the same sign as the aggregate resistance temperature coefficient of the contact.

There is no need to discuss here the method of plotting the curves, nor is there any necessity for plotting the curves accurately. Therefore in the figures only the general shape of each curve is intended to be conveyed, and though the non-physical branches have been inserted, isolated ovals have all been omitted. Asymptotes are shown by dotted straight lines. It is convenient to divide the discussion into cases delimited by the signs of the constants a and c. The reader may be reminded that the curves ought to be sheared in the manner already described before being compared with experimental curves.

Case I.—a Positive, c Positive.—The curves take the forms shown in Figs. 16A, 16B, 16C, according as b/ρ is less than, equal to, or greater than a/c, *i.e.*, $2\gamma(P-P')$ less than, equal to, or greater than $\sigma-\sigma'$. The greatest asymmetry is indicated in Fig. 16A, which corresponds to the condition that the Thomson coefficient shall be great compared with that of the Peltier effect and of the temperature coefficient of increase of resistance.

Case II.—a Positive, c Negative.—The existence of coherer action brings about two distinct sub-cases which both indicate that the contact would operate well as a detector for radio-telegraphy. In one of the cases the contact would be a sensitive detector only when the E.M.F. is negative; in the other it would be sensitive at two unequal and opposite values of the applied E.M.F. The analytical condition separating these cases is cumbrous (Figs. 17A and 17B).

Many samples of galena form with lead contacts of these types.

Case III.—a Negative, c Positive.—There is only one type of curve. A contact belonging to this class will behave as a sensitive detector only for positive applied E.M.F. (Fig. 18).

Case IV.—a Negative, c Negative.—There are four sub-cases shown in Figs. 19A, 19B, 19C, 19D. In A, b/ρ is much smaller than a/c—that is to say, $2\gamma(P-P') << \sigma-\sigma'$. In B, $2\gamma(P-P')$ is slightly smaller than $\sigma-\sigma'$. A case scarcely different from the last in its physical branch is given in *c*, where $2\gamma(P-P')=\sigma-\sigma'$. Finally, in D is seen the case when $2\gamma(P-P')>\sigma-\sigma'$. If we regard the passage through these sub-cases as being due to a gradual increase of the share of coherer action in the whole phenomenon, it is evident that coherer action always tends to make the realisable part of the curve symmetrical about the origin of co-ordinates. If, on the other hand, we regard the transition as due to gradual elevation of the neutral temperature, we see cases A to D correspond to higher and higher values of this temperature.

Contacts including zincite usually fall in one of these sub-cases.

Most of the curves drawn in this Paper possess a vertica tangent on one side of the origin of co-ordinates, and many o them a vertical tangent at each side. After allowing for the shearing process required to convert the curves into "characteristic curves," it may be concluded broadly that when a is positive the current produced by a definite voltage is greater when the voltage is negative than when it is positive; while, when a is negative or zero the positive voltage produces the greater current. Now it is well known that when a contact is warmed by direct communication of heat, the thermoelectric force is for some pairs of substances in *same* direction as the larger, and for other pairs in the *opposite* direction to the larger of the currents produced across the contact by reversing a constant applied E.M.F. The latter case has frequently been quoted as showing that these contacts did not owe their rectifying properties to thermoelectric effects. This is a fallacy. For the observation of the direction of the thermoelectric force produced by direct heating often shows only the sign of the Peltier effect and not the sign of the Thomson effect in the circuit; and it is the latter alone which decides, through the constant a of this Paper, the accordance or disaccordance indicated above.

Considerations relative to the Determination of the Constants involved in the Theory.

The constants occurring in the general steady current equation are a, b, c and ρ. By aid of the equations of the section dealing with the thermoelectric theory, two of these constants may be expressed partially in terms of the more fundamental constants appearing in the thermo E.M.F. equation

$$e_\theta = \theta(\beta + 2\alpha z + \alpha\theta)$$
$$= \theta(\beta' + \alpha\theta), \text{ say.}$$

Here θ is the temperature of the contact above that of the cold junction and the surroundings z. We thus obtain

$$a = \frac{\alpha}{m^2}, \quad b = \frac{\beta'}{m}.$$

The quantity m is the measure of the rate of loss of heat by radiation and conduction.

Now, for any particular two substances forming a contact the constants α and β' may be determined by measurement of the thermoelectric force for two values of θ, and thus a and b become known if m can be determined. Unfortunately this quantity cannot be determined. Of the other constants in the current equation—namely c and ρ—the first is given by the equation

$$c = \rho\gamma/m.$$

The coefficient γ is, for the crystals that form the most interesting contacts, very difficult to measure. In all the methods the author has employed, difficulty has arisen through the high thermoelectric effect at the electrodes; but so far as the measurements go it is negative in many conducting native oxides and sulphides. The quantity ρ can be roughly estimated by shearing an experimental curve till its asymptote is parallel to the axis of y and then noticing that the tangent at the origin is $x - \rho y = 0$.

The data accumulated up to the present by the author will be published in a later communication, when also the experimental work of other observers will be discussed.

APPENDIX.

On the Operation of a Contact as a Radio-telegraphic Detector.

It appears that the earliest statement of the general connection between the steady current curve of a contact and its pro-

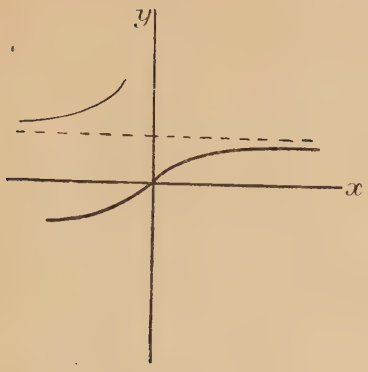


FIG. 4.

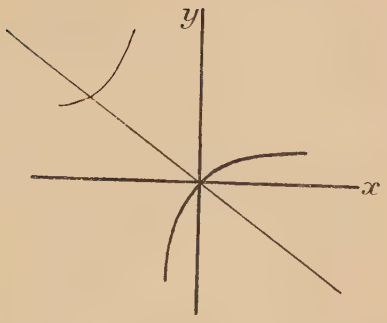


FIG. 5.

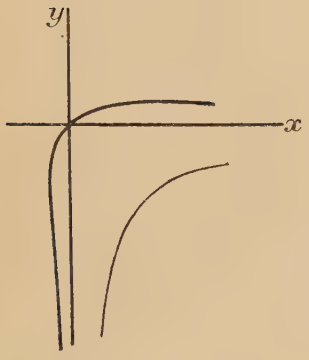


FIG. 7.

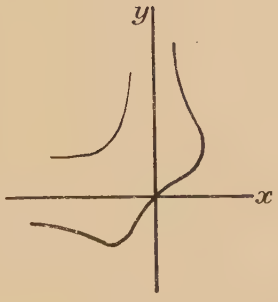


FIG. 8.

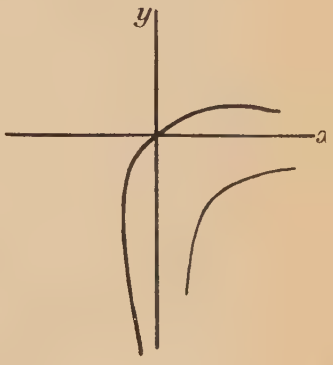


FIG. 11.

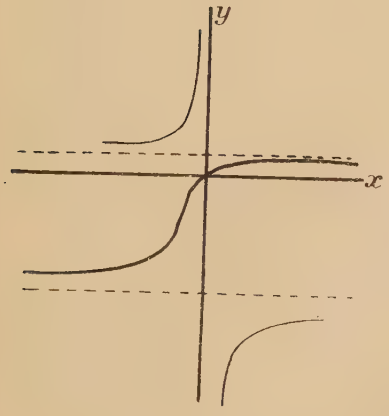


FIG. 13.

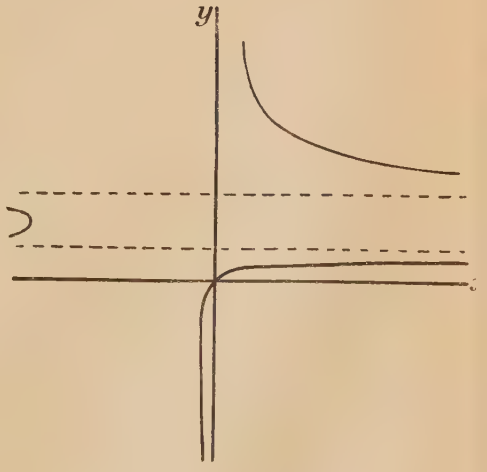


FIG. 14.

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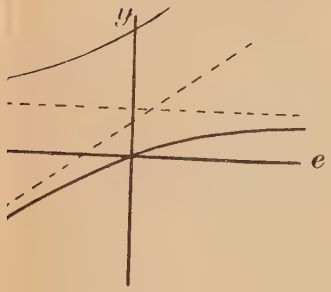


Fig. 6.

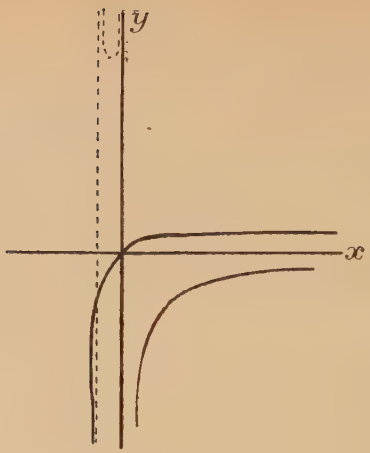


Fig. 16A.

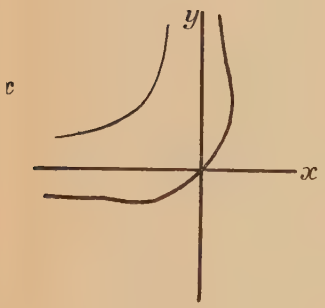


Fig. 12.

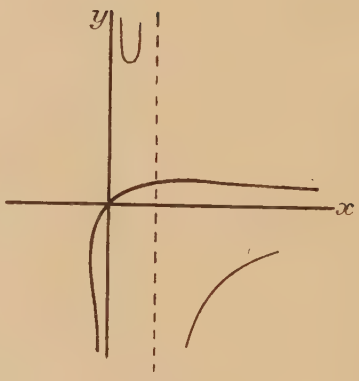


Fig. 17A.

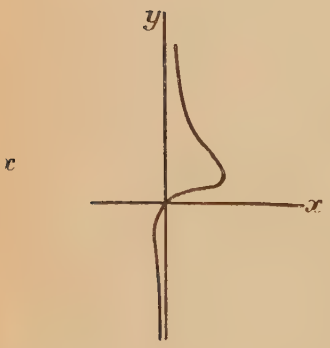


Fig. 15.

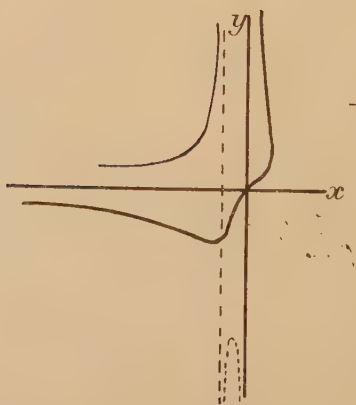
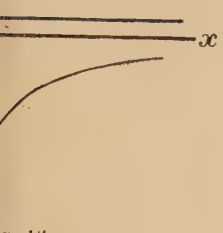


Fig. 19A.



G. 16B

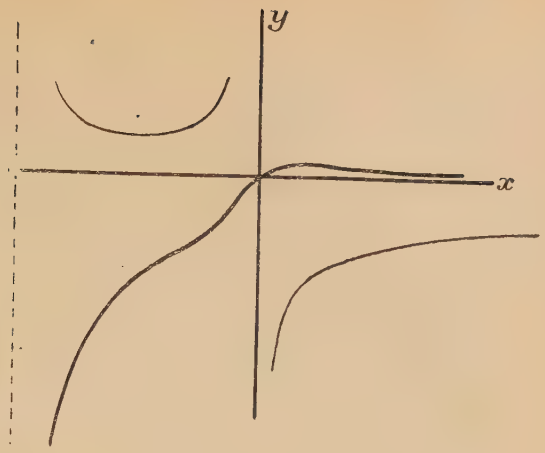
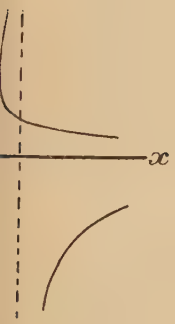


FIG. 16C



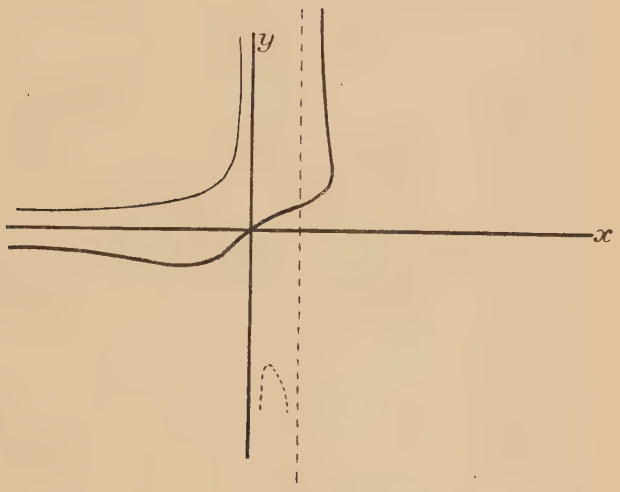


FIG. 18.

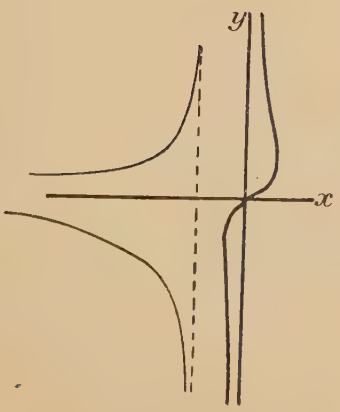


FIG. 19C.

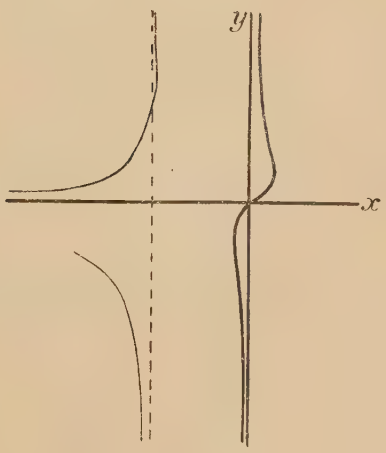


FIG. 19D.

perty of acting as a detector of oscillations was made by H. Brandes* in 1906. He pointed out that any kind of conductor —not necessarily a contact—which departs from Ohm's law will rectify alternating current in some degree. If with such a substance (or contact) the volt-ampere curve is symmetrical in the first and third quadrants, a direct current of suitable magnitude must be kept flowing and the oscillatory current to be rectified must be superposed on this current. On the other hand, he pointed out, if the characteristic curve is not symmetrical about the origin, no direct E.M.F. is actually needed. And he showed that the same general rules hold good for all sorts of detectors, such as the electrolytic, as well as for the so-called crystal rectifiers. Again, Pickard† has indicated that the greatest sensitiveness of crystal contacts is at the E.M.F. where the rate of change of resistance is greatest, and J. A. Fleming‡ has published diagrams showing that in the case of his vacuum valve the E.M.F. of maximum sensitiveness is near that of the maximum ordinate of the second derived curve of the characteristic. These principles may now be regarded as well known and have been tacitly assumed in the preceding pages. It would be well, however, to examine the matter from the purely theoretical standpoint of this Paper.

Let the equation of the characteristic of any detector (Fig. 20) be $y=f(e)$, where y is the current produced by applying the steady E.M.F. e to the detector. Suppose the E.M.F. to increase by the small amount h and the current to increase in consequence by the amount k. Then, by Taylor's theorem,

$$k=hf'(e)+\frac{h^2}{\underline{2}}f''(e)+\frac{h^3}{\underline{3}}f'''(e)+,$$

where f', f'', f''' ... indicate the first, second, third ... derived functions of $f(e)$. We will suppose that h is a very small alternating E.M.F., say, $h=a \sin \omega t$, where ω is small.

The whole matter may now be regarded from two points of view. In the first place we may concentrate attention on the rectifying action of the contact, and calculate the excess of electricity passing in one direction round the circuit over that passing in the other direction, when the alternating E.M.F. is acting. In the second place we may endeavour to estimate,

* H. Brandes, "E.T.Z.," 27, p. 1015, Nov. 1, 1906.
† G. W. Pickard, "El World," November, 1906.
‡ J. A. Fleming, Royal Institution Discourse, June 4, 1909, and British Patent Specification 13518, 1908.

not what may be called the rectified current, but, instead, the energy possessed by the portion of the current in the circuit due to the harmonic E.M.F.

In the former case the average value of the increased current flowing in the circuit while the alternating E.M.F. is acting is

$$\delta y = \frac{1}{T}\int_0^T k\,dt,$$

where T is the period of the alternation and $T = 2\pi/\omega$. Hence

$$\delta y = \frac{a^2}{2\lfloor 2}f''(e) + \frac{3a^4}{8\lfloor 4}f^{iv}(e) + \ldots.$$

Thus the magnitude of the average rate of flow of the rectified

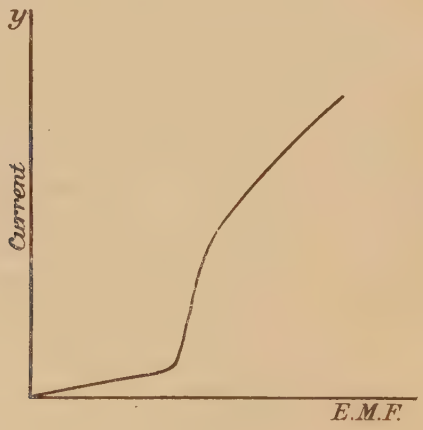


Fig. 20.

electricity depends on the derived functions of even orders as might be expected. The equation shows that if the steadily applied E.M.F. e is adjusted to a value corresponding to a sharp bend on the characteristic curve, then the rectified current may be very large, according to the value of $f''(e)$ principally. It is on this account that a coherer with a perfectly symmetrical characteristic can be made to act as a rectifier of slow alternating current.

On the other hand, from the energy point of view, the increase of the mean square value of the current round the

circuit, when alternating E.M.F. is imposed on the steady E.M.F., must be calculated. This increase is given by

$$(y+k)^2-y^2=\left\{f(e)+hf'(e)+\frac{h^2}{2}.f''(e)+\right\}^2-\{f(e)\}^2$$
$$=2hf(e).f'(e)+h^2[\{f'(e)\}^2+f(e).f''(e)]$$
$$+h^3[\tfrac{1}{3}f(e)f'''(e)+f'(e).f''(e)+]+\ldots.$$

Thus, the additional work done in the circuit per second, in consequence of the imposition of the harmonic E.M.F. h, is a function of the gradient of the characteristic as well as of the higher derived functions. In particular, work will be done in the circuit even when using a straight portion of the characteristic curve where all the derived functions of even order might be zero, notwithstanding that in this case the rectified current would be zero. The apparent paradox that work may be done in the telephone circuit although there is no rectified current is removed by reflecting that a sine wave alternating current will not deflect a direct-current galvanometer though it may be doing work in the circuit.

We therefore draw the conclusion that when the steady E.M.F. is adjusted to a sharp bend on the characteristic curve the detector gives the best rectification ; but when the greatest deflection of an instrument of hot-wire type is required, we must have the quantity $\{f'(e)\}^2+f(e).f''(e)$ near its maximum, which will usually occur very near the E.M.F. corresponding to the point of inflexion on the curve.* In this latter case the mode of operation may be styled the "mean square mode." In general the amplitude of the oscillatory E.M.F. will perhaps not be so minute as is here assumed, and consequently the local steady E.M.F. applied to the detector cannot be so adjusted as to separate the two modes of action just described.

The above analysis leads immediately to an expression for the efficiency of transformation of energy. The instantaneous rate of working of the applied oscillatory E.M.F. is hy, and thus, if W be the average rate of working,

$$W=\frac{1}{T}\int_0^T hydt$$
$$=\frac{a^2}{2}f'(e)+\frac{a^4}{16}f'''(e)+.$$

* In the case of the magnetic detector it has been shown experimentally by the author that the point of maximum sensitiveness is near, but not quite coincident with, the point of inflexion of the hysteresis loop.—"Proc. Phys. Soc.," Vol. XX., June, 1906.

The average rate of working in the circuit of the detector is

$$w=m\,.\,\frac{a^2}{2}[\{f'(e)\}^2+f(e)\,.\,f''(e)]+m\,.\,\frac{a^4}{32}[3\{f''(e)\}^2 +4f'(e)\,.\,f'''(e)+f(e)\,.\,f^{\mathrm{iv}}(e)]+\ldots,$$

where m is a proportional constant. The efficiency is the ratio w : W.

The degree of the equation connecting w and W when a is eliminated between the above equations depends on the number of terms in powers of a^2 included in the process. The simplest case occurs when w is near its maximum and also

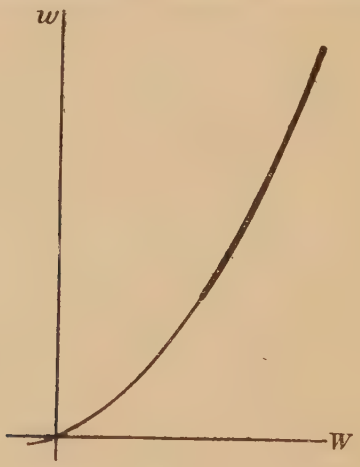


Fig. 21.

$f'''(e)=0$; for near the maximum of w the coefficient of a^4 in the equation for w vanishes, and then we have

$$wf'(e)=m[\{f'(e)\}^2+f(e)f''(e)]\mathrm{W}.$$

When W and w are taken as abscissæ and ordinates this equation connecting input and output of energy is represented by a straight line through the origin. More generally, writing the equation for w in the form

$$w=\alpha_1 a^2+\alpha_2 a^4$$

and

$$\mathrm{W}=\tfrac{1}{2}f'(e)a^2+\tfrac{1}{16}f'''(e)\,.\,a^4,$$

we have

$$\{\alpha_2\mathrm{W}-\tfrac{1}{16}f'''(e)w\}^2=\{\tfrac{1}{2}f'(e)\,.\,\alpha_2-\tfrac{1}{16}f'''(e)\,.\,\alpha_1\}\times\{\tfrac{1}{2}f'(e)w-\alpha_1\mathrm{W}\},$$

which represents a parabolic locus.

In the cases in which $f'''(e)$ and higher derived functions may

be neglected—and these are often practical cases—this equation becomes of the form

$$4a_2(W+A)^2=f'(e)(w+B),$$

where A and B are positive constants involving $f(e)$, $f'(e)$ and $f''(e)$. The locus is shown in Fig. 21. The thick portion is that with which radio-telegraphy is concerned, for elsewhere W is negative or it is so small as to give no audible sound. Experimental observations made by the author on the magnetic detector,* the coherer,† the electrolytic detector and various contact rectifiers† have yielded points lying approximately on a straight line passing to the right of the origin in the manner suggested by a straight line drawn through the thickened part of the last figure.

From the experimental results mentioned, it would appear that this theory of the " detecting " property of any apparatus with a characteristic curve like Fig. 20, under low-frequency E.M.F. of small amplitude, holds good in some measure for the high frequencies and large amplitudes employed by the various observers.

ABSTRACT.

The Paper is a purely theoretical one, and deduces mathematically the laws connecting the current and the applied E.M.F. in a circuit containing a light contact of two conductors. When an electric current passes across a light contact of two different substances, heat is liberated or absorbed in accordance with the law of Peltier, heat is generated in accordance with the law of Joule, and, in the regions of the conductors where there is a temperature gradient, heat is liberated or absorbed in accordance with the laws of the Thomson effect. These thermal actions are very noticeable in contacts made of badly conducting natural oxides or sulphides on account of the high resistivity and the large thermoelectric effects in these substances. The low thermal conductivities of these substances exalt the electrical consequences by conserving the heat. The bulk of the wireless telegraphy of the world is carried on by such contacts as these, and the present Paper, therefore, constitutes a theory of the action of these detectors.

The general equation connecting the current across a contact and the P.D. between two points of the circuit is deduced, and is found to be of fourth degree when the difference of the specific heats of electricity in the conductors is not zero. The curves are discussed and plotted for all the principal variations of value of the Thomson effects, the Peltier effects, and the resistivity-temperature coefficients of the substances. It is shown that, in general, when a contact is used as a radiotelegraph detector, there may be some coherer

* " Proc. Phys. Soc.," Vol. XX., 1906.
† Vol. XXII., 1910.

action mixed up with the thermoelectric "rectifying" action; and that, whether there is or is not any coherer action, the principal features of the characteristic curve connecting current and E.M.F. are determined by the Thomson rather than by the Peltier effect. Thus, if two contacts could be made having all their electrical and thermal properties of equal measure excepting the Thomson property, of which the measures were of opposite signs, the curve connecting current and E.M.F. for one contact would rise more rapidly on, say, the side of positive E.M.F. than on the side of negative E.M.F., and vice versa for the other contact; and this although a direct experiment of applying heat to the contacts in turn and observing the consequent thermoelectric forces, showed these to have the same direction in both.

DISCUSSION.

Dr. A. Russell considered Dr. Eccles' fundamental equation very valuable. He thought Dr. Eccles' definition of the efficiency of a detector useful and the method of arriving at it very neat, but he did not think the curve of efficiency given by Dr. Eccles would be much use. What was wanted to tell a good detector was the shape of the function showing the relation between the current and the E.M.F.

Prof. C. H. Lees thought it was important to have shown that the well-known phenomena of thermoelectricity and variation of resistance could explain the behaviour of contacts. The theory was quite elastic enough to explain more complicated characteristic curves than those now considered, as the resistance was not strictly a linear function of the temperature and the heat conductivity also was not a constant.

Mr. P. R. Coursey referred to the tests he had recently carried out for Dr. Fleming on the subject of crystal detectors, some of the results of which were before the meeting, and said they were undertaken more with the view to finding the relation (if any) that exists between the characteristic and sensitiveness curves for the crystals (taking impressed voltages as abcissæ in both cases) than to discovering a reason for the shape of the characteristic curves and the more or less abrupt changes of curvature, or "kinks," that occur in them, as had been done by Dr. Eccles in his Paper. It had been known for some time past that when the second differential of the characteristic-curve of a Fleming oscillation valve is plotted it bears a considerable resemblance to the sensitiveness curve of the valve. This relation was found in the above tests to hold good to a greater or less degree in the majority of crystal detectors, the most notable exception being the galena-plumbago combination, for which practically no agreement between the two curves could be found. Similar remarks applied to the electrolytic detector which was also tested; and hence one was led to the conclusion that in most detectors we have at least two actions taking place tending to produce sounds in the telephones, and that these may or may not assist one another, it being probable that in the galena detector the effect due to the changes of curvature are almost entirely masked by some other, and opposing, effect due to a different cause—perhaps some type of electrolytic or thermal action. Hence to analyse or explain the shape of the characteristic curve of a detector in the manner outlined by Dr. Eccles was not necessarily going to be of assistance in finding a good detector, since, as seen from the above tests, the sensitiveness is not always dependent merely upon the shape of the characteristic curve, although nevertheless in many cases it will doubtless lead to valuable results.

Dr. J. A. Fleming communicated the following: With reference to Dr. Eccles' remarks on the characteristic curves of certain rectifying detectors, I believe I was the first to point out, in a British patent specification, No. 13,518 of 1908, the conditions under which certain rectifiers

of oscillations, such as my glow-lamp detector, can be operated with an added E.M.F. as a more sensitive detector than by use as a mere rectifier without such E.M.F. Long before that date a large number of measurements of characteristic curves of my valves and other detectors had shown that there were sudden changes of curvature in them. The method of detecting these was to plot a first and second differential curve from the volt-ampere characteristic. In the above specification I described simple means of applying the boosting voltage in the case of a glow-lamp detector by using a potentiometer wire in connection with the same battery which incandesces the glow lamp. With certain detectors this is a more sensitive arrangement for detecting oscillations, but it is unfortunately as yet impossible to construct glow-lamp detectors or oscillation valves with predetermined characteristics. I have, however, one glow-lamp detector which when worked in this manner is more sensitive than any electric wave detector I have ever met. We have not yet found the way to reproduce it. I referred to these experiments in a Friday evening discourse at the Royal Institution, and exhibited a diagram showing the characteristic curve of a certain valve and its first and second differential curves. The scheme of the circuits and the method of working is fully explained with diagrams on p. 481 of Chapter VI. of the second edition of my book on "The Principles of Electric Wave Telegraphy and Telephony." Since that date a very large amount of information has been collected in my laboratory on the form of the characteristic curves of a number of contact and crystal detectors. Mr. Coursey made for me a very extensive collection of such measurements about eight or nine months ago, but this information and a numerous set of curves depicting the observations has never yet been published. The results do not, however, fall in with any very simple theory of such rectifying detectors. The point to be explained is the reason for this sudden change in curvature in the characteristic. It is obvious that if there is such a change of curvature then the mean value of the current through the detector and, therefore, in a telephone, in series with it, will be increased if we add to the alternating E.M.F. acting on the detector a steady unidirectional E.M.F. of value exactly corresponding to the abscissa of the characteristic at that point of change of curvature.

The Author, in reply to Prof. Lees, stated that all the coefficients of his fundamental equation were variable with temperature. In reply to Prof. Fleming, he stated that the Paper fully explains the main changes of curvature in the characteristic of a crystal detector, but does not touch on the theory of the vacuum valve.

XXXII.—*On the Evaluation of Certain Combinations of the Ber, Bei and Allied Functions.* *By* S. BUTTERWORTH, *M.Sc., Assistant Lecturer in Physics, School of Technology, Manchester.*

RECEIVED MAY 8, 1913. READ MAY 30, 1913.

1. It has been pointed out by Russell* that, in most formulæ, the ber and bei functions nearly always occur in one or other of the following combinations :—

$$\begin{aligned} \mathrm{X}\ (x) &= \mathrm{ber}^2 x + \mathrm{bei}^2 x, \\ \mathrm{V}\ (x) &= \mathrm{ber}'^2 x + \mathrm{bei}'^2 x, \\ \mathrm{Z}\ (x) &= \mathrm{ber}\ x\ \mathrm{ber}' x + \mathrm{bei}\ x\ \mathrm{bei}' x, \\ \mathrm{W}\ (x) &= \mathrm{ber}\ x\ \mathrm{bei}' x - \mathrm{bei}\ x\ \mathrm{ber}' x, \end{aligned}$$

while in most cases only the combinations Z/X, W/X, Z/V, W/V occur.

Russell has given formulæ for evaluating these combinations, suitable either for high or low values of the argument. These formulæ have been extended by Savidge† and he has published a four-figure table of the functions. A five-figure table of the combinations Z/V and W/V for smaller intervals of the argument has also been prepared by Rosa and Grover.‡

In the present Paper formulæ are established for functions φ_n and ψ_n defined by the relation

$$\varphi_n + \iota\psi_n = \frac{\mathrm{J}_{n+1}(\sqrt{-\iota}x)}{\mathrm{J}_{n-1}(\sqrt{-\iota}x)}.$$

From the definition of the ber and bei functions it may readily be shown that

$$\left.\begin{aligned} \mathrm{W/X} &= x(1+\varphi_1)/2, & \mathrm{W/V} &= 2/x - x\psi_2/4 \\ \mathrm{Z/X} &= -x\psi_1/2, & \mathrm{Z/V} &= x(1+\varphi_2)/4 \end{aligned}\right\}, \quad \ldots \quad (1)$$

so that Russell's formulæ are particular cases of the formulæ of this Paper.

The functions φ_n and ψ_n also occur in the problem of a spherical conductor in an alternating magnetic field, n in this case being of the form $(2m+1)/2$, where m is an integer.

* Russell, "Phil. Mag.," 6, 17, p. 524, 1909.
† Savidge, "Phil. Mag.," 6, 19, p. 49, 1910.
‡ Rosa and Grover, "Bulletin" Bureau of Standards, p. 226, Jan., 1912.

2. *Formulæ for Low Values of* x.—By writing down the series for $J_{+}(x)$ and $J_{n-1}(x)$, and dividing, afterwards putting $\sqrt{-\iota x}$ for x, we get

$$\left.\begin{aligned}\varphi_n &= -\frac{2x^4}{(2n)^2(2n+2)(2n+4)} + \frac{(28n+48)x^8}{(2n)^4(2n+2)^2(2n+4)(2n+6)(2n+8)} \\ \psi_n &= -\frac{x^2}{2n(2n+2)} + \frac{(10n+12)x^6}{(2n)^3(2n+2)^2(2n+4)(2n+6)}\end{aligned}\right\} \text{(A)}$$

These formulæ are useful only for very small values of x.

3. *Formulæ for High Values of* x.—By making use of the relation

$$J_n(\iota x) = \iota^n I_n(x),$$

and using the semi-convergent expansion

$$I_n(x) = \frac{1}{\sqrt{2\pi x}} e_x \left\{1 - \frac{4n^2-1}{8x} + \frac{(4n^2-1)(4n^2-9)}{2(8x)^2} - \dots\right\},$$

we get

$$\left.\begin{aligned}\varphi_n &= -1 + \frac{2n}{\sqrt{2}.x} - \frac{2n(2n-1)(2n-3)}{8\sqrt{2}x^3} \\ &\quad - \frac{2n(2n-1)(2n-3)}{8x^4} + \frac{2n(2n-1)(4n^2-9)(2n-7)}{\sqrt{2}.128x^5} \\ \psi_n &= -\frac{2n}{\sqrt{2}x} + \frac{2n(2n-1)}{2x^2} - \frac{2n(2n-1)(2n-3)}{8\sqrt{2}x^3} \\ &\quad - \frac{2n(2n-1)(4n^2-9)(2n-7)}{\sqrt{2}.128x^5}\end{aligned}\right\} \text{(B)}$$

Putting $n=1$ or 2, and making use of relation (1), Russell's formulæ for W/X, W/V, Z/X, Z/V are readily obtained.

4. *Formulæ for Intermediate Values of* x.—It is found on application that formulæ (A) and (B) fail for a considerable range of the argument.

If, however, the roots of $J_{n-1}(x)=0$ are known, this gap may be filled up.

Let $\alpha_1, \alpha_2, \alpha_3 \dots$ be the positive roots of $J_{n-1}(x)=0$.

Then we may write

$$J_{n-1}(x) = a_n(x^2-\alpha_1^2)(x^2-\alpha_2^2)(x^2-\alpha_3^2)\dots.$$

Since the series for $J_{n+1}(x)$ is more convergent than that for

$J_{n-1}(x)$, $J_{n+1}(x)/J_{n-1}(x)$ may be separated into partial fractions by the usual method, and we find

$$\frac{J_{n+1}}{J_{n-1}}=4nx^2\sum_{r=1}^{\infty}\frac{1}{a_r^2(a_r^2-x^2)}. \quad \ldots\ldots \quad (2)$$

Putting $\sqrt{-\iota}x$ for x,

$$\left.\begin{aligned}\varphi_n&=-4nx^4\sum_{r=1}^{\infty}\frac{1}{a_r^2(a_r^4+x^4)}\\ \psi_n&=-4nx^2\sum_{r=1}^{\infty}\frac{1}{a_r^4+x^4}\end{aligned}\right\} \quad \ldots\ldots \quad (3)$$

These formulæ are always convergent, but the convergence is very slow.

However, by expanding the later terms in powers of x, practical formulæ may be obtained.

For values of x up to $x=a_2$, I find the most convenient form to use is

$$\left.\begin{aligned}\varphi_n&=-4nx^4\left\{\frac{1}{a_1^2(a_1^4+x^4)}+\frac{1}{a_2^2(a_2^4+x^4)}+C_3\right.\\ &\qquad\left.-C_5x^4+C_7x^8-\ldots\right\}\\ \psi_n&=-4nx^2\left\{\frac{1}{a_1^4+x^4}+\frac{1}{a_2^4+x^4}+C_2-C_4x^4+C_6x^8-\ldots\right\}\end{aligned}\right\}, \quad (C)$$

where

$$C_s=\sum_{r=3}^{\infty}\frac{1}{a_r^s},$$

When $x>a_2$ formulæ (B) are suitable.

The coefficients C_s are easily determined by summation, except in the case of C_2. However, by comparing the coefficients of x^2 in formulæ (A) and (C) we find

$$C_2+\frac{1}{a_1^4}+\frac{1}{a_2^4}=\frac{1}{8n^2(2n+2)},$$

and this relation is sufficiently accurate to determine C_2.

5. *Numerical Values from Formulæ* (C).—The zeroes of $J_0(x)$ and $J_1(x)$ are given to a high degree of accuracy in Gray and Mathews' Bessel Function (pp. 244 and 280). From these values I obtain the values of the coefficients given in Table I. In Table II. are given the values of W/X, Z/X (*a*) as calculated by formulæ (C), using relation (1), (*b*) as calculated by Savidge.

The values of the argument chosen are those for which the calculation by the usual formulæ is most tedious.

In Table III. are the corresponding values of W/V and Z/V as compared with Rosa and Grover's values.

Numerical values from formulæ (B) are unnecessary, as these reduce to Russell's formulæ when $n=1$ or 2.

TABLE I.

s.	C_s (for $n=1$).	C_s (for $n=2$).
2	$2{\cdot}7338\times10^{-4}$	$1{\cdot}56456\times10^{-4}$
3	$2{\cdot}8933\times10^{-6}$	$1{\cdot}16285\times10^{-6}$
4	$3{\cdot}5003\times10^{-8}$	$9{\cdot}9744\times10^{-9}$
5	$4{\cdot}4599\times10^{-10}$	$9{\cdot}0709\times10^{-11}$
6	$5{\cdot}8173\times10^{-12}$	$8{\cdot}4836\times10^{-13}$
7	$7{\cdot}6745\times10^{-14}$	$8{\cdot}0504\times10^{-15}$
8	$1{\cdot}0183\times10^{-15}$	$7{\cdot}7129\times10^{-1}$
9	$1{\cdot}3552\times10^{-1}$	$7{\cdot}3960\times10^{-19}$
10	$1{\cdot}8065\times10^{-19}$	$7{\cdot}1222\times10^{-21}$
a_1	2·4048256	3·8317060
a_2	5·5200781	7·0155867

TABLE II.

x	W/X.		Z/X.	
	Savidge.	Formula (C).	Savidge.	Formula (C).
3	0·7485	0·748531	0·5399	0·539948
4	0·7141	0·714078	0·5842	0·584194
5	0·7101	0·710145	0·6040	0·603998
6	0·7101	0·710139	0·6211	0·621090

TABLE III.

x	W/V.		Z/V.	
	Rosa and Grover.	Formula (C).	Rosa and Grover.	Formula (C).
5	0·81709	0·817090	0·69496	0·694960
6	0·79786	0·797862	0·69781	0·697809
7	0·78377	0·783768	0·70037	0·700372
8	0·77361	0·773610	0·70214	0·702130

ABSTRACT.

Formulæ are obtained for the real and imaginary parts of J_{n+1}/J_{n-1}, when the argument is of the form $\sqrt{-\iota}\,x$. These formulæ are useful for evaluating the functions W/X, Z/X, W/V, Z/V defined by Russell, and have other applications.

The formulæ are of two types :—

(*a*) Those suitable for low or moderate values of the argument. These are particularly useful in those cases where computation by other methods is most laborious.

(*b*) Those suitable for high values of the argument. Russell's formluæ are particular cases of these formulæ.

XXXIII. *The Extraordinary Ray resulting from the Internal Reflection of an Extraordinary Ray at the Surface of an Uniaxal Crystal.* *By* JAMES WALKER, *M.A., Oxford.*

RECEIVED MAY 8TH, 1913. READ MAY 30, 1913.

TAKING one of the co-ordinate axes in the direction of the optic axis of the crystal, the equation of the extraordinary wave-surface may be written

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $b=c$, or $c=a$, or $a=b$, according as it is the axis of x, or of y, or of z that is along the optic axis, and the ray-slowness s in the direction (λ, μ, ν) is given by

$$s^2=\frac{\lambda^2}{a^2}+\frac{\mu^2}{b^2}+\frac{\nu^2}{c^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If, then, (p, q, r) be the direction-cosines of the normal to the reflecting surface and the suffixes $(_1)$, $(_2)$ refer to the incident and the reflected extraordinary rays respectively, the principle of least time gives the equations

$$\frac{1}{s_2}\left(\frac{\lambda_2}{a^2}, \frac{\mu_2}{b^2}, \frac{\nu_2}{c^2}\right)+\frac{1}{s_1}\left(\frac{\lambda_1}{a^2}, \frac{\mu_1}{b^2}, \frac{\nu_1}{c^2}\right)=\mathrm{K}(p, q, r), \quad . \quad (3)$$

where K is an undetermined multiplier.

Introducing in place of (p, q, r) the direction-cosines (α, β, γ) of the diameter of the wave-surface that is conjugate to the plane $px+qy+rz=0$, equations (3) become

$$\frac{1}{s_2}(\lambda_2, \mu_2, \nu_2)+\frac{1}{s_1}(\lambda_1, \mu_1, \nu_1)=\mathrm{L}(\alpha, \beta, \gamma), \quad . \quad . \quad . \quad (4)$$

where

$$\mathrm{L}=\mathrm{K}\Big/\sqrt{\frac{\alpha^2}{a^4}+\frac{\beta^2}{b^4}+\frac{\gamma^2}{c^4}}.$$

1. As a first result we obtain

$$\begin{vmatrix} \lambda_2, & \lambda_1, & \alpha \\ \mu_2, & \mu_1, & \beta \\ \nu_2, & \nu_1, & \gamma \end{vmatrix}=0,$$

or the diameter of the extraordinary wave-surface, described round the point of incidence, that is conjugate to its section by the reflecting surface, is co-planar with the incident and the reflected extraordinary rays.

2. Let $(\alpha', \beta', \gamma')$ be the direction-cosines of a line in this plane perpendicular to (α, β, γ), then, multiplying equations (4) by α', β', γ' respectively and adding, we obtain

$$\frac{1}{s_2} \sin \chi_2 = \frac{1}{s_1} \sin \chi_1,$$

where χ_1, χ_2 are the angles that the direction (α, β, γ) makes with the incident and the reflected rays.

Since the ray-velocities are given by the radii vectores of the wave-surface, this equation expresses that the diameter of the extraordinary wave-surface, described round the point of incidence, that is conjugate to its section by the reflecting surface, is the median of the triangle formed by the incident and the reflected extraordinary rays and a parallel to the reflecting surface.

3. Again multiplying equations (4) by p, q, r respectively and adding, we obtain

$$\frac{1}{s_2} \cos \varphi_2 + \frac{1}{s_1} \cos \varphi_1 = \mathrm{L} \cos \theta,$$

where φ_1, φ_2, θ are the angles that the normal to the reflecting surface makes with the incident and the reflected rays and the direction (α, β, γ).

Now $\frac{1}{s_1} \cos \varphi_1$, $\frac{1}{s_2} \cos \varphi_2$ are the projections on the normal to the reflecting surface of the ray-velocities along the incident and the reflected rays, and from the above results these are equal. Hence

$$\mathrm{L} = \frac{2}{s_1} \frac{\cos \varphi_1}{\cos \theta}$$

and

$$\frac{\lambda_2}{2\alpha \cos \varphi_1 - \lambda_1 \cos \theta} = \frac{\mu_2}{2\beta \cos \varphi_1 - \mu_1 \cos \theta} = \frac{\nu_2}{2\gamma \cos \varphi_1 - \nu_1 \cos \theta}$$

$$= \frac{\cos \varphi_2}{\cos \varphi_1 \cos \theta} = \frac{1}{\sqrt{4 \cos^2 \varphi_1 - 4 \cos \chi_1 \cos \varphi_1 \cos \theta + \cos^2 \theta}}$$

From the form of these equations it follows that they hold for any system of rectangular co-ordinates.

ABSTRACT.

By the principle of least time it is shown that the diameter of the extraordinary wave-surface described round the point of incidence, that is, conjugate to the reflecting surface, is coplanar with the incident and reflected extraordinary rays and is the median of the triangle formed by these rays and a parallel to the reflecting surface.

The direction-cosines of the reflected ray are then obtained in terms of those of the incident ray and the said diameter of the wave-surface.

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